No further reproduction or distribution of this copy is permitted by electronic transmission or any other means.

The user should review the copyright notice on the following scanned image(s) contained in the original work from which this electronic copy was made.

Section 108: United States Copyright Law

The copyright law of the United States [Title 17, United States Code] governs the making of photocopies or other reproduction of copyrighted materials.

Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the reproduction is not to be used for any purpose other than private study, scholarship, or research. If a user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of "fair use," that use may be liable for copyright infringement.

This institution reserves the right to refuse to accept a copying order if, in its judgement, fulfillment of the order would involve violation of copyright law. No further reproduction and distribution of this copy is permitted by transmission or any other means.
On Generic Global Rigidity

ROBERT CONNELLY

Abstract. It is shown that the conditions, generic redundant rigidity and 4-connectivity, are not sufficient for a graph to be globally rigid in three-space at a generic configuration. The example is $K_{5,5}$.

1. Introduction

Given a collection of points in some Euclidean space and the distances between some pairs of the points, when can we find another collection of points that have the same distances between corresponding points? Of course, we can simply move the whole collection by a rigid congruence, but when is there another essentially different configuration? This is what we call the question of global rigidity, and we regard it as a fundamental geometric problem.

Conditions for global rigidity, that is conditions for unique graph embeddings (see Hendrickson [7]), are related to the existence of molecular conformations (see Crippen and Havel [4] as well as the rigidity of frameworks and the structural stability of frameworks.

Here we investigate the problem of global rigidity when the configuration of points is in "generic" position. Roughly speaking, this means that the points are "randomly" chosen so that no special constraints are satisfied. In [7] it is conjectured that certain other well-known conditions on the graph are equivalent to this generic global rigidity. We show in dimension three that for at least one graph, the complete bipartite graph $K_{5,5}$, Hendrickson's conditions are not sufficient.

The author gratefully thanks Bruce Hendrickson, Maria Terrel, Bob Walker, and all three referees for their many helpful comments and corrections.

2. Basic definitions

2.1. A framework. Let $G$ be a finite graph (with no loops or multiple edges) with $v$ vertices, and let $(p_1, p_2, \ldots, p_v)$ be a corresponding set of
\( v \) points in \( \mathbb{R}^d \). Note that these points are not necessarily distinct. We call this labeled collection of points \((\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_v) = \mathbf{p}\) a configuration in \( \mathbb{R}^d \). For every edge in \( G \) we imagine that there is a rigid bar joining the corresponding points of the configuration. The graph \( G \) together with the configuration \( \mathbf{p} \) is denoted by \( G(\mathbf{p}) \) and is called a framework in \( \mathbb{R}^d \). We often denote such a framework as in Figure 2.1.1. Note that the bars of the framework are allowed to overlap without necessarily having a vertex at the intersection.

![Figure 2.1.1](image)

**Figure 2.1.1**

### 2.2. Equivalent framework in \( \mathbb{R}^d \) and global rigidity.

We say that two frameworks \( G(\mathbf{p}) \) and \( G(\mathbf{q}) \) are equivalent, written as \( G(\mathbf{p}) \cong G(\mathbf{q}) \), if whenever \( \{i, j\} \) corresponds to an edge of \( G \), then

\[
|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|,
\]

where \( |\mathbf{p}| \) is the Euclidean norm of the vector \( \mathbf{p} \). In other words, corresponding bars of the two frameworks have the same length.

Note, however, that there is a very simple way to obtain equivalent frameworks. Namely we can take the configuration \( \mathbf{p} \) and apply any fixed rigid congruence of \( \mathbb{R}^d \) to each point \( \mathbf{p}_i \) to get \( \mathbf{q}_i \) and thus another configuration \( \mathbf{q} \). In this case we say that \( \mathbf{p} \) is congruent to \( \mathbf{q} \), written as \( \mathbf{p} \cong \mathbf{q} \).

We say that \( G(\mathbf{p}) \) is globally rigid in \( \mathbb{R}^d \) if for any configuration \( \mathbf{q} \), \( G(\mathbf{q}) \cong G(\mathbf{p}) \) implies that \( \mathbf{p} \cong \mathbf{q} \). In other words, when \( G(\mathbf{p}) \) is globally rigid, only congruent configurations give equivalent frameworks. For example, in \( \mathbb{R}^2 \) Figure 2.2.1(a) is equivalent to Figure 2.2.1(b), but they are not congruent. Figure 2.2.1(c) is globally rigid in any \( \mathbb{R}^d \) for \( d \geq 2 \).

![Figure 2.2.1](image)

**Figure 2.2.1**
2.3. Generic configurations. Consider any configuration \( p \) in \( \mathbb{R}^d \), but regard \( p \) as a point in \( \mathbb{R}^{d_u} \) by looking at all \( d_u \) of its coordinates. We say that \( p \) is a generic configuration if all of its \( d_u \) coordinates are algebraically independent over the rationals, or, in other words, if the coordinates of \( p \) do not satisfy any nonzero polynomial equation with integer coefficients.

If a configuration is generic, its points cannot have any undue symmetries and they cannot satisfy any “special” conditions. Whether the configuration \( p \) is generic or not can influence whether \( G(p) \) is globally rigid. For example, the framework in Figure 2.3.1(a) is not globally rigid in \( \mathbb{R}^2 \). However, when the same graph is chosen but with a generic configuration, as in Figure 2.3.1(b), then it is globally rigid.

![Figure 2.3.1](image)

Figure 2.3.1

Our basic problem then is to determine, for a given graph \( G \), whether \( G(p) \) is globally rigid for all generic configurations \( p \). This also brings up the following question, first raised by Maria Terrell: Is global rigidity a generic property? In other words, if \( G(p) \) is globally rigid for one generic configuration \( p \), is \( G(q) \) globally rigid for all other generic configurations \( q \)? The answer seems to be yes, but we do not have a proof. See §3.2, where a related concept “generic rigidity” is discussed.

3. Related basic results and definitions

3.1. Infinitesimal rigidity. An important form of rigidity is a linearization known as infinitesimal rigidity. Let \( G(p) \) be a framework in \( \mathbb{R}^d \). Let \( p' = (p'_1, p'_2, \ldots, p'_u) \) be another configuration in \( \mathbb{R}^d \) but regarded as a vector \( p'_i \), a “velocity”, associated to each point \( p_i \). We say that \( p' \) is an infinitesimal flex or an infinitesimal motion if for all bars \( \{i, j\} \) in \( G \), the following holds:

\[
(p_i - p_j)(p'_i - p'_j) = 0,
\]

where the product of vectors is the usual scalar product.

However, for any configuration \( p \) there are certain infinitesimal flexes that always exist. These trivial infinitesimal flexes can be described as the derivative at time 0 of a path of congruences of \( \mathbb{R}^d \) (starting at the identity),
or explicitly $p'_i = Sp_i + t$, where $S$ is a $d$ by $d$ skew symmetric matrix and $t$ is a vector fixed and independent of $i$.

If a framework $G(p)$ has only trivial infinitesimal flexes, then we say $G(p)$ is infinitesimally rigid in $\mathbb{R}^d$. For example, the frameworks of Figures 3.1.1(a) and 3.1.1(b) are infinitesimally rigid in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and the frameworks of Figures 3.1.1(c) and 3.1.1(d) are not infinitesimally rigid in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The four top vertices of Figure 3.1.1(d) are coplanar. The infinitesimal motion or velocities are indicated on Figures 3.1.1.(c) and 3.1.1.(d).

![Figure 3.1.1](image)

3.2. Generic rigidity. If $G(p)$ is infinitesimally rigid for one configuration $p$, then $G(q)$ may or may not be infinitesimally rigid for another configuration $q$. Nevertheless, there is a "typical" behavior for a given graph $G$. We say that a graph $G$ is generically rigid in $\mathbb{R}^d$ if for some configuration $p$, $G(p)$ is infinitesimally rigid in $\mathbb{R}^d$. The reason we call this "generic" rigidity is the following basic result from algebraic geometry. (See Gluck [6], Asimow and Roth [2], or Connelly [5] for a proof.)

**Theorem 3.2.1.** If the graph $G$ is generically rigid in $\mathbb{R}^d$ and $p$ is any generic configuration, then $G(p)$ is infinitesimally rigid in $\mathbb{R}^d$.

By virtue of generic rigidity being a generic property for configurations, it is only a property of the graph $G$ and not any particular configuration $p$. See Sugihara [9], Lovász and Yemini [8], or Hendrickson [7] for a deterministic polynomial time complexity algorithm that decides when a graph $G$ is generically rigid in $\mathbb{R}^2$. For $\mathbb{R}^3$ no such algorithm is known, and indeed, the search for such an algorithm is a well-known unsolved problem.
3.3. Graph connectivity. We say that a graph $G$ is vertex $n$-connected or just $n$-connected if one needs to remove at least $n$ vertices from $G$ to disconnect the remaining vertices of $G$. For example, the graph of Figure 3.3.1(a) is 3-connected but not 4-connected. The graph of Figure 3.3.1(b) is 2-connected but not 3-connected.

![Figure 3.3.1](image)

Note that if $G(p)$ is to be globally rigid in $\mathbb{R}^d$, even when $p$ is generic, $G$ must be at least $(d + 1)$-connected. For example, the framework represented by Figure 3.3.1(b) cannot be globally rigid in $\mathbb{R}^2$, since we can reflect part of the vertices of $G(p)$ about the line separating the vertices. This idea works in any dimension.

4. Hendrickson's conjecture

How does one determine when a framework $G(p)$ is globally rigid in $\mathbb{R}^d$ at a generic configuration $p$? We saw in §3.3 that $(d + 1)$-connectivity was necessary. The following result of Hendrickson [7] provides more conditions.

We say a graph $G$ is generically redundantly rigid in $\mathbb{R}^d$ if for each bar of $G$, $G$ minus that bar is generically rigid in $\mathbb{R}^d$. The complete graph on $n$ vertices is the graph with an edge between all pairs of $n$ vertices.

**Theorem 4.1 (Hendrickson).** If a framework $G(p)$ is globally rigid in $\mathbb{R}^d$ for all generic configurations $p$, then either $G$ is the complete graph with $d + 1$ or fewer vertices, or the following conditions hold:

(i) the graph $G$ is $(d + 1)$-connected, and
(ii) the graph $G$ is generically redundantly rigid.

Hendrickson also conjectured that these two conditions are sufficient.

**Conjecture (Hendrickson).** If a graph $G$ is generically redundantly rigid in $\mathbb{R}^d$ and $G$ is $(d + 1)$-connected, then for all generic configurations $p$, $G(p)$ is globally rigid in $\mathbb{R}^d$.

Hendrickson's Conjecture is easily seen to be true for $d = 1$. It looks very likely to be true for $d = 2$ as well. It will be shown that it is false for $d \geq 3$. 

5. Review of results for bipartite graphs: Bolker-Roth theory

A complete bipartite graph is a finite graph, in which the vertices are partitioned into two sets $A$ and $B$ with edges joining every vertex of $A$ with every vertex of $B$ and no other edges. If $A$ has $n$ vertices and $B$ has $m$ vertices, we call this finite graph $K_{m,n}$.

In Bolker and Roth [3] there is a very thorough treatment of conditions which are equivalent to $K_{m,n}(p)$ being infinitesimally rigid. We will implicitly use their results by means of the following interpretation, an alternate point of view presented in Whiteley [10].

**Theorem 5.1 (Whiteley [10]).** The complete bipartite framework $K_{m,n}(p)$ for $m, n \geq 2$, $d \geq 2$ in $\mathbb{R}^d$ has a nontrivial infinitesimal motion if and only if either

(i) the points of $A \cup B$ lie on a quadric surface, or

(ii) one side $(A$ or $B$) lies on a hyperplane along with at least one point of the other side, or

(iii) one side $(A$ or $B$) lies on a hyperplane $H$ and lies on a quadric surface within $H$.

We will concentrate on condition (i). Suppose that the quadric surface $S$ is given by

$$S = \{ p_i \in \mathbb{R}^d | (p_i)' Q p_i = 1 \},$$

where $Q$ is a symmetric $d$ by $d$ matrix, and $(\cdot)'$ is the transpose operation. Then Whiteley gives a very simple description of the nontrivial infinitesimal motion $\mathbf{p}'$. It is given by

$$p_i' = \begin{cases} Qp_i & \text{if } i \text{ is in } A, \\ -Qp_i & \text{if } i \text{ is in } B. \end{cases}$$

See Figure 5.1 for a picture of the infinitesimal motion in case $Q$ is the identity matrix and $S$ is the unit sphere. It is easy to check that it is nontrivial when $m, n \geq 2$, the points are distinct, and $Q$ is nonsingular. The arrows represent the infinitesimal flex. We will call this infinitesimal flex the inny-outy flex.

![Figure 5.1](image-url)
6. The averaging principle

The following is a result relating infinitesimal rigidity and global rigidity. Proofs, which are easy, can be found in Connelly [5].

**Theorem 6.1.** Suppose that \( p' \) is an infinitesimal flex for a framework \( G(p) \) in \( \mathbb{R}^d \), where the points of \( p \) do not all lie in a hyperplane. Then

\[
G(p + p') = G(p - p'),
\]

and \( p + p' \) is congruent to \( p - p' \) if and only if \( p' \) is a trivial infinitesimal flex.

For example, in \( \mathbb{R}^2 \) the framework of Figure 6.1(a) is equivalent to the framework of Figure 6.1(c) using the infinitesimal flex of the framework of Figure 6.1(b). Only one vertex has a nonzero infinitesimal motion.

![Figure 6.1](image)

7. The example

7.1. The main theorem. We consider the case \( K_{5,5}(p) \) in \( \mathbb{R}^3 \). A unique quadric surface is determined by nine generic points in \( \mathbb{R}^3 \). Thus when the ten points of \( K_{5,5}(p) \) form a generic configuration, they do not lie on a quadric surface and condition (i) of Whiteley’s Theorem 5.1 does not hold. Note that five generic points alone do not lie in a hyperplane, and so conditions (ii) and (iii) do not hold either. Thus \( K_{5,5} \) is generically rigid in \( \mathbb{R}^3 \). Furthermore, \( K_{5,5} \) has ten vertices and 25 edges. Thus there are 25 linear equations (3.1) for infinitesimal rigidity, with 30 variables, the coordinates of \( p' \). There is a six-dimensional space of solutions to these equations corresponding to the space of trivial infinitesimal flexes. So one of the equations corresponding to one of the edges is redundant. Thus \( K_{5,5}(p) \) is infinitesimally rigid even after the removal of one of its edges. But the symmetries of the abstract graph \( K_{5,5} \) are transitive on the edges. Thus, since \( p \) is a generic configuration, \( K_{5,5}(p) \) is infinitesimally rigid after the removal of any edge. Thus \( K_{5,5}(p) \) is redundantly rigid.
It is also easy to check that $K_{5,5}$ is at least 4-connected. (Indeed, it is 5-connected.) Thus Hendrickson's conjecture says that $K_{5,5}(p)$ should be globally rigid for all generic configurations $p$. We will show on the contrary the following:

**Theorem 7.1.1.** There is a generic configuration $p$ in $\mathbb{R}^3$, where $K_{5,5}(p)$ is not globally rigid.

Before we proceed with the proof, we define the following subset of the space of all configurations:

$$\text{IF}(G) = \{p | G(p) \text{ is not infinitesimally rigid in } \mathbb{R}^d \},$$

the space of infinitesimally flexible configurations in $\mathbb{R}^d$ for $G$. Note that the space of all configurations, flexible or not, is $du$-dimensional. We regard this as $\mathbb{R}^{du}$.

For the case of $K_{5,5}$ for $\mathbb{R}^3$ we see that $du = 30$, and conditions (i), (ii), and (iii) define the subset $\text{IF}(K_{5,5})$ of $\mathbb{R}^{30}$. Since the nine points $p_1, p_2, \ldots, p_9$ can be chosen arbitrarily, as well as two of the three coordinates of $p_{10}$, we can choose 29 coordinates freely (in some open set) and have the points lie on a quadric surface. Thus, condition (i) of Theorem 5.1 defines a 29-dimensional subset (an algebraic set) of $\mathbb{R}^{30}$. The other two conditions define a lower-dimensional subset. In any case, $\text{IF}(K_{5,5})$ is 29-dimensional.

The following lemma is critical for showing that our example is a counter-example to Hendrickson's conjecture.

**Lemma 7.1.2.** There exists a configuration of ten points $(p_1, p_2, \ldots, p_{10}) = p$ on the unit sphere in $\mathbb{R}^3$ and a $t > 0$ such that

$$p + tp' \notin \text{IF}(K_{5,5}),$$

where $p'$ is Whiteley's inny-outy flex for $K_{5,5}(p)$ in $\mathbb{R}^3$.

**Proof.** The most direct way to do this problem seems to be to find ten points on the unit sphere and then use equations (3.1) to solve for the infinitesimal motions for $t = 1/2$, say. Of course, the $p$ used in (3.1) is $p + tp'$ here. We omit this calculation, which amounts to solving 25 linear equations in 30 unknowns. (Alternatively, this calculation is equivalent to computing the rank of a 25 by 30 matrix.)

**7.2. The proof of the main result.** Consider the function

$$g: \text{IF}(K_{5,5}) \times \mathbb{R} \to \mathbb{R}^{30}$$

defined by $g(p, t) = p + tp'$, where $p'$ is Whiteley's inny-outy infinitesimal flex for $K_{5,5}(p)$ defined in §5. The domain of $g$ is only the set of those $(p, t)$ where the configuration $p$ determines a unique quadric surface. This is an open subset of $\text{IF}(K_{5,5}) \times \mathbb{R}$. We know that $g$ is well defined on this subset by using condition (i) of Theorem 5.1, since that is where $p'$
is well defined and unique. In any case, the domain and range of $g$ is a 30-dimensional manifold, and $g$ is a smooth algebraic function.

We will show that the image of $g$ contains a generic point. By considering the case when $t = 0$, we see that the image of $g$ contains $IF(K_{5,5})$. The smooth algebraic image of a smooth manifold near a generic point is a smooth manifold again. By Lemma 7.1.2, we know that the image of $g$ is not in $IF(K_{5,5})$. Since being infinitesimally rigid is a generic property, it is also clear that the image of $g$ is not in the intersection of $IF(K_{5,5})$ and any neighborhood of any point in $IF(K_{5,5})$. Thus the image of $g$ must be greater than 29-dimensional. The image must be 30-dimensional and hence must contain an open set of $\mathbb{R}^{30}$ and a generic point of $\mathbb{R}^{30}$.

Suppose that $p + tp'$ is this generic configuration. By the averaging principle, Theorem 6.1, $K_{5,5}(p + tp')$ is equivalent to $K_{5,5}(p - tp')$. Hence, $K_{5,5}(p + tp')$ is not globally rigid at a generic configuration.

7.3. Remarks and generalizations. In $\mathbb{R}^3$, $K_{5,5}$ is the only example we know which is a counter-example to Hendrickson's Conjecture. In $\mathbb{R}^d$ for $d \geq 4$, an argument similar to the above (which we omit) shows the following:

**Theorem 7.3.1.** For $m, n \geq d + 2$ and $m + n = (d + 1)(d + 2)/2$, there is a generic configuration $p$ in $\mathbb{R}^d$ such that $K_{m,n}(p)$ is not globally rigid.

It is easy to check that $K_{m,n}(p)$ as above is also $(d + 1)$-connected and redundantly rigid in $\mathbb{R}^d$. Thus each of the above graphs serve as a counter-example to Hendrickson's Conjecture.

References


Department of Mathematics, Cornell University, Ithaca, New York 14853

E-mail address: connelly@mssun7.msi.cornell.edu