Review: Models of $H^2$:

(Poincaré) Disk

Half-plane

Hyperboloid

\[ (x^2 + y^2 - z^2 = -1) \]
\[ z > 0 \]

\[ ds^2 = dx^2 + dy^2 - dz^2 \]

(geodesics, isometric, geodesics)

which is positive definite when restricted to the hyperboloid.

Symmetries are Möbius transformations fixing the boundary (and preserving the boundary (and its orientation)).

For the half-plane: $PSL_2(C\mathbb{R})$.

One isometry from the hyperboloid to the disk model is given by stereographic projection centered at the point $(0,0,-1)$:

\[ (x,y,z) \text{ in the hyperboloid} \]
\[ \text{maps to} \quad \left( \frac{x}{z+1}, \frac{y}{z+1} \right) \text{ in the disk}. \]

This is analogous to the usual stereographic projection of a sphere onto a plane.
Ideal Polygons and their Moduli

Note that the hyperbolic plane $\mathbb{H}^2$ has an ideal boundary.

In the disk model or half-plane model, the ideal boundary consists of the points of the boundary circle or boundary line (including $\infty$), respectively.

$\overline{\mathbb{H}^2}$ denotes $\mathbb{H}^2$ with the ideal boundary added. (This is not a metric in a natural way, since the ideal points are infinitely far away.)

Equivalently, and more intrinsically, we can think of the points of the ideal boundary as equivalence classes of rays, where two rays $r_1$ and $r_2$ are equivalent if

$$\lim_{t \to \infty} \text{dist}(r_1(t), r_2(t)) = 0,$$

that is, $r_1$ and $r_2$ are asymptotically close together.

We want all these rays to be equivalent:

To see that this makes sense, suppose we have two rays tending toward the same ideal boundary point.

By symmetry, we may assume this ideal boundary point is $\infty$ in the half-plane model.
Now we have this configuration:
\[ r_1(t) \uparrow \quad r_2(t) \]

We can upper-bound the hyperbolic distance from \( r_1(t) \) to \( r_2(t) \) by integrating the hyperbolic metric along the Euclidean line segment from \( r_1(t) \) to \( r_2(t) \). But the hyperbolic metric is proportional to the Euclidean metric divided by \( y \).

As \( t \to \infty \), \( y \to \infty \) and the Euclidean length of this segment stays constant, so the hyperbolic arclength of this segment goes to zero.

Consequently, the ideal boundary of \( \mathbb{H}^2 \) can be given an intrinsic definition in terms of the metric of \( \mathbb{H}^2 \).

**Triangles** A triangle in \( \mathbb{H}^2 \) is a region bounded by three geodesics.

Letting \( \alpha, \beta, \gamma \) denote the three angles, it turns out that \( \alpha + \beta + \gamma < \pi \) (unlike in the Euclidean case where \( \alpha + \beta + \gamma \) would equal \( \pi \)).

And in fact, \[
\text{Area of Triangle} = \pi - (\alpha + \beta + \gamma) \]
This can be proved from the Gauss-Bonnet Theorem: The total turning of a curve bounding a disk $D$ in a Riemannian manifold is

$$2\pi - \int_D K dA$$

where $K$ is the curvature (a scalar in two dimensions), and $dA$ is the area form from the metric.

When $K$ is negative, the curve turns more than would be expected.

**Exercise** Measure the total curvature of some surface (e.g. a leaf) by cutting a thin strip along the boundary, laying the strip flat on a piece of paper, and measuring the total turning. Can you find a positive-curvature leaf?

**Getting the area of a sphere from Gauss-Bonnet:** apply the theorem to a hemisphere. The boundary is a great circle which does not turn at all.

So

$$0 = 2\pi - \int_{\text{Hemisphere}} K dA.$$  But $K = 1$,

So

$$2\pi = \int_{\text{Hemisphere}} dA = \text{Area (Hemisphere)}.$$  Thus a hemisphere has area $2\pi$ and the whole sphere has area $4\pi$. 

In the case of hyperbolic triangles, all the turning happens at the corners.

Since $K = -1$,

$$(\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi + \text{Area}.$$  
Subtracting $2\pi$ from both sides proves the formula

$$\text{Area} = \pi - (\alpha + \beta + \gamma),$$
and in particular, $\alpha + \beta + \gamma < \pi$.

Interesting fact: since $\alpha + \beta + \gamma > 0$, $\text{Area} < \pi$. Every hyperbolic triangle has area less than $\pi$.

As you move the three corners of a hyperbolic triangle towards the ideal boundary, the limiting shape

is the region bounded by three geodesics tending towards three distinct ideal boundary points. This is called an ideal triangle.

Examples:

The angles at the corners are all $0$, so by the area formula, the area of an ideal triangle is exactly $\pi$. 
Ideal triangles turn out to be much simpler than ordinary triangles.

**Proposition.** Any two ideal triangles are related by an (essentially unique) isometry.

**Proof.** Consider the half-plane model for \( \mathbb{H}^2 \). The ideal boundary is \( \mathbb{R}P^1 = \mathbb{R} \cup \{ \infty \} \), and the symmetry group \( \text{PSL}_2(\mathbb{R}) \) acts on \( \mathbb{R}P^1 \) in the obvious way.

A basic fact says that any three distinct points in \( \mathbb{R}P^1 \) can be taken to any other three distinct points by a (unique) projective transformation in \( \text{PGL}_2(\mathbb{R}) \).

For example, if we want to send \( a, b, c \) to \( 0, 1, \infty \), respectively, the unique map is

\[
x \mapsto \frac{(x-a)(b-c)}{(x-c)(b-a)}
\]

Now \( \text{PSL}_2(\mathbb{R}) \) is index 2 in \( \text{PGL}_2(\mathbb{R}) \) and \( \text{PSL}_2(\mathbb{R}) \) corresponds to only the orientation-preserving isometries of \( \mathbb{H}^2 \). These two complications luckily cancel each other out, it turns out.

So the moduli space of ideal triangles is zero-dimensional.
What about ideal quadrilaterals, i.e., things like \(\text{ }\) or \(\text{ }\) (but not \(\text{ }\))

The area of such an ideal quadrilateral is \(2\pi\), by Gauss-Bonnet or by triangulating the quadrilateral into two ideal triangles.

Are all ideal quadrilaterals equivalent by isometries? \(\text{No}\).

If we try to send an ideal quadrilateral \(Q_1\) to an ideal quadrilateral \(Q_2\), we can send the first three vertices of \(Q_1\) to their counterparts in \(Q_2\). But then the isometry is locked and we have no freedom of where to send the fourth vertex.

This yields an invariant of ideal quadrilaterals. Given four points \(a, b, c, d\) on the ideal boundary, send \(a, b, c\) to \(0, 1, \infty\) and see where \(d\) goes.

By the previous page, \(d\) goes to

\[
\frac{(d-a)(b-c)}{(d-c)(b-a)}
\]

This is called the cross-ratio.

So the moduli space of ideal quadrilaterals is 1-dimensional.
More generally, the moduli space of ideal \( n \)-gons has dimension

\[
\dim \text{PSL}_2(\mathbb{R}) = \frac{n(n-3)}{2}
\]

in choosing the \( n \) vertices.

There's something funny about this fact. topic given two ordinary (non-ideal) hyperbolic or Euclidean triangles, there is at most one way to join them along designated edges:

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\]

must become

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\]

because you must align the endpoints of the two edges and this then determines the configuration. So the moduli space of ordinary quadrilaterals can be understood in terms of the moduli space of ordinary triangles.

This doesn't work for ideal triangles and polygons, because the edges have no endpoints to align. In fact, you can slide one of the ideal triangles along its edge, while keeping the other triangle fixed.

To measure this sliding, note that an edge of an ideal triangle has a unique
"center" that is equidistant from the other two edges:

When gluing two ideal triangles along an edge, the two centers need not line up, but can be a distance $\delta$ apart.

This parameter $\delta$ is what accounts for the degree of freedom in the moduli space of ideal quadrilaterals.

We would like things a lot better, though, if we could describe our ideal polygons by just specifying an ordinary "length" to each edge (in a triangulation of the polygon).

Since the actual lengths will always be infinite, one needs to instead renormalize. A key ingredient is the notion of a horocycle.

Circles in $H^2$ A circle is the set of points at a fixed distance from a fixed point (the center of the circle).
Claim: Hyperbolic circles in the disk model or half-plane model appear as circles.

For example, in the disk model, circles look like

\[
\begin{array}{c}
\text{\includegraphics{circle.png}}
\end{array}
\]

Proof: If the circle is centered at the origin in the disk model, this is clear by rotational symmetry about the origin. In the disk model, if the center is not the origin, we can find an isometry which moves the center to the origin (the middle of the disk), since \( \mathbb{H}^2 \) is homogeneous. This isometry is represented by a Möbius transformation, and Möbius transformations preserve circles, so we are done. For the half-plane model, note that the isometry from the disk \( \mathbb{H}^2 \) to the half-plane \( \mathbb{H} \) is given by a Möbius transformation (which preserves circles).

\[
\begin{array}{c}
\text{\includegraphics{circle.png}}
\end{array}
\]

However, centers of circles are not preserved by Möbius transformations.

In the half-plane model, the center of this circle is very close to the boundary.
A horocycle is the limit of a set of circles (through a fixed point) as the center moves to the ideal boundary.

In the halfplane and Poincaré disk models, a horocycle looks like a circle tangent to the ideal boundary.

Equivalently, a horocycle is a "circle" centered at an ideal point.

Horocycles have the property that any line through the "center" is perpendicular to the horocycle, just like an ordinary circle.

Exercise. Find a natural geometric meaning for a circular arc in the disk or half-plane models, which curves (with respect to the hyperbolic metric) but not enough to be a horocycle or proper circle, like the following:

Hint: find the right model and put the interesting points in convenient places.
Decorated ideal polygons:

A decorated ideal polygon is an ideal polygon with a choice of a horocycle around each vertex:

(The horocycles are allowed to overlap)

If a side or diagonal of a decorated ideal polygon, we can now define a renormalized length \( l(A) \) by measuring the distance along the edge between the two horocycles.

If the horocycles overlap, these lengths can be negative.

Lemma: Any three choices for \( l(A), l(B), l(C) \) can be realized by a decorated ideal triangle, unique up to symmetry.

Proof: All ideal triangles look the same, so we might as well be working with this one.
Choose a random decoration and measure the renormalized lengths, \( l(A)_0 \), \( l(B)_0 \), and \( l(C)_0 \).

Now consider what happens when we "decrease the radius" of the three horocycles by \( \alpha, \beta, \gamma \) (possibly negative).

The new renormalized lengths are:

\[
\begin{align*}
l(A) &= l(A)_0 + \beta + \gamma \\
l(B) &= l(B)_0 + \gamma + \alpha \\
l(C) &= l(C)_0 + \alpha + \beta.
\end{align*}
\]

Clearly there is a unique choice of \( (\alpha, \beta, \gamma) \) to yield each possibility for \( (l(A), l(B), l(C)) \). Q.E.D.
Aside: The case where $l(A) = l(B) = l(C)$
looks like the configuration
with three circles all tangent
to each other and to the ideal boundary.

Up to Möbius transformations, there's a
unique configuration of four pairwise-tangent circles.
Apparently this is discussed in the book
*Indra's Pearls* by Mumford et al.

Anyhow, for decorated ideal polygons,
the glueing ambiguity has been
eliminated. For example, given two decorated ideal triangles with matching renormalized side lengths, there is a unique way to glue them to get a decorated ideal quadrilateral

So, a decorated ideal quadrilateral is uniquely described by the renormalized lengths of the four sides and one of the diagonals.

More generally,
**Theorem:** Given a decorated ideal polygon
and a triangulation, the renormalized lengths of the edges in the triangulation determine the decorated polygon uniquely.

So, the space of decorated ideal $n$-gons, modulo $\text{PSL}_2(\mathbb{R})$,
looks like $\mathbb{R}^{2n-3}$. (There are $2n-3$ edges in the triangulation.)