Curves on Surfaces: Lecture 4

September 13, 2012 (notetaker: Adam Kalman)

Today we will discuss the hyperbolic plane.

**Definition 1.** The hyperbolic plane is the unique complete, simply connected Riemannian surface with curvature $-1$.

This definition is nice, but nearly useless for computation, so we will develop 3 different models that will allow us to get a handle on this surface. The first one is the Poincare disk model.

**The Poincare Disk Model**

We start with a disk of radius 1 (equation $x^2 + y^2 < 1$), and impose the Riemannian metric

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - r^2)^2},$$

where $r^2 = x^2 + y^2$. So for instance, arc length is

$$\int \frac{ds}{dt} = 2 \int \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{1 - r^2}. $$

Note that this metric is conformal, since at every point, this metric is just a multiple of the standard Euclidean metric $ds^2 = dx^2 + dy^2$. This means that angles agree with Euclidean angles, so when drawing a picture, distances may be distorted but angles are not.

![Figure 1: Poincare disk model](image)
After going through a lot of analysis of arc length, you can figure out what the geodesics are. The geodesics turn out to be circular arcs that are perpendicular to the boundary (or a line segment if it goes through the center point of the disk). Note that there is a unique geodesic between any two points, and some other familiar axioms of plane geometry are true as well. Also note that the sum of the angles of a triangle in this model is strictly less than 180 degrees, essentially because the sides of a triangle in this model are bent inward.

In this model, the area of a triangle is equal to \( \pi - \text{(angle sum)} \). One way to see why this is true is to check that the angle sum defect \( (\pi - \text{(angle sum)}) \) is an additive measure on triangles (i.e. if we take two triangles that are right next to each other to form a larger triangle, the sum of the angle sum defects of the two smaller triangles is the angle sum defect of the larger triangle they form).

Another closely related model is the Poincare Half-Space Model.

The Poincare Half-Space Model

Here we start with the upper half-plane \( \mathbb{H}^2 = \{(x, y) : y > 0\} \). You can work out what metric you need to get constant curvature \(-1\), and it turns out to be

\[
    ds^2 = \frac{dx^2 + dy^2}{y^2}.
\]

Again, this is conformal, and the geodesics are semicircles perpendicular to the boundary (a limiting circle is a vertical line).

![Figure 2: Poincare half-plane model](image)

Why is this the right metric and why are the geodesics just semicircles? Better than trying to use Riemannian geometry or do some sort of variational problem to find what the geodesics are, the much easier thing to do is to look at the automorphisms (more correctly, isometries) of the disk or half-plane.

Analyzing these models using Mobius Transformations (LFTs)

What can the automorphisms be? Note that these automorphisms are conformal (i.e. angle-preserving) because our metric is a standard linear multiple of the Euclidean metric. This is equivalent to saying it's a complex analytic map (locally invertible) with nonzero derivative. What possible complex analytic automorphisms might you have? A natural class of complex...
analytic automorphisms of the disk is the group of Mobius transformations (or linear fractional transformations, LFTs). These are maps from \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of the form \( z \mapsto \frac{az+b}{cz+d} \) (where \( \hat{\mathbb{C}} \) is the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \), so that this map is now well-defined at infinity).

An amazing fact is that coefficients of LFTs (viewed as matrices like \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)) compose like matrix multiplication. Specifically, if the map above is composed with the map \( z \mapsto \frac{a'z+b'}{c'z+d'} \), we get another LFT, whose four coefficients are given by the entries of the matrix resulting from the multiplication \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \). To see that this is true, think of \( \hat{\mathbb{C}} \) as \( \mathbb{CP}^1 \) (pairs of complex numbers modulo scaling). We can usually scale so that the second coefficient is just 1, but if the second coefficient is zero, we get the extra point \( \frac{1}{0} \) added to the plane. Now, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a z_1 + b z_2 \\ c z_1 + d z_2 \end{pmatrix} \), and if we scale this vector so that the second coefficient is 1, we get \( \begin{pmatrix} a z_1 + b z_2 \\ c z_1 + d z_2 \end{pmatrix} \), so we see that this is really just \( \text{PGL}(2, \mathbb{C}) \) acting on \( \mathbb{CP}^1 \).

For some other amazing properties of LFTs, such as the fact that they preserve circles, watch the video "Mobius Transformations Revealed" at [http://www.youtube.com/watch?v=OzlfIsUNh04](http://www.youtube.com/watch?v=OzlfIsUNh04).

Let’s think about what certain simple LFTs do geometrically. In the case where the denominator of an LFT is 1, we just get the map \( z \mapsto a z + b \), which is just translations, scalings, and rotations. If the LFT is just \( z \mapsto \frac{1}{z} \), this is just complex conjugation \( (z \mapsto \overline{z}) \) composed with circle inversion \( (z \mapsto \frac{1}{z}) \). These two cases \( (z \mapsto a z + b \text{ and } z \mapsto \frac{1}{z}) \) generate all LFTs (analogously, the two corresponding 2-by-2 matrices generate \( \text{PGL}(2, \mathbb{C}) \)).

We also want the complex analytic maps we are seeking (as possible automorphisms of the disk or upper half-plane) to preserve the disk or upper half-plane, respectively. Here is the important theorem to this end:

**Theorem 1.** The group of orientation-preserving isometries of the upper half-plane, \( \text{Isom}^+(\mathbb{H}^2) \), is exactly \( \text{PSL}(2, \mathbb{R}) \), sitting inside of \( \text{PGL}(2, \mathbb{C}) \), the group of all LFTs.

**Proof.** Because the hyperbolic plane (thought of as the upper half-plane \( \mathbb{H}^2 \)) has constant curvature \(-1\), \( \mathbb{H}^2 \) is homogeneous and isotropic, so \( \mathbb{H}^2 \) must have an isometry taking every point to every other, and every direction to every other. Also, every isometry here is conformal as a map from \( \mathbb{H}^2 \) to itself, and furthermore it extends to a map from \( \hat{\mathbb{C}} \) to itself by the Schwarz reflection principle. Since it’s a conformal automorphism of \( \hat{\mathbb{C}} \), it must be an LFT. Which LFT must it be? It must preserve the real axis, so it must be in \( \text{PGL}(2, \mathbb{R}) \). Now, why don’t we get all of \( \text{PGL}(2, \mathbb{R}) \)? There are a few groups to consider here: \( \text{PSL}(2, \mathbb{R}) \), \( \text{PGL}(2, \mathbb{R}) \), \( \text{PSL}(2, \mathbb{C}) \), and \( \text{PGL}(2, \mathbb{C}) \). There is really no difference between \( \text{PSL}(2, \mathbb{C}) \), and \( \text{PGL}(2, \mathbb{C}) \), since we can always rescale an element of \( \text{PGL}(2, \mathbb{C}) \) by dividing by the square root of the determinant, and then once we projectivize, we get an element of \( \text{PSL}(2, \mathbb{C}) \). But with real matrices \( \text{PGL}(2, \mathbb{R}) \), we get 2 connected components (determinant greater than 0 and determinant less than 0), and \( \text{PSL}(2, \mathbb{R}) \) can be thought of as those matrices with determinant \(-1\). But also, the LFT has to preserve the real axis, so consider some matrix with determinant \(-1\) and interpret it as a conformal isometry. For example, consider \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \). This is just \( z \mapsto -z \), which is a rotation of the plane by 180 degrees about the origin. It takes the upper half-plane to the lower half-plane. We want to
do a little better than this - things not just in $\text{PGL}(2, \mathbb{R})$ but in $\text{PSL}(2, \mathbb{R})$ (things with positive determinant).

And in fact, you could go backwards. To get the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$, we could assume it’s invariant under $\text{PSL}(2, \mathbb{R})$, and that it’s just $ds^2 = dx^2 + dy^2$ at the [conveniently chosen] point $(x = 0, y = 1)$. We need to find some automorphism that takes $(0, 1)$ to any $(x, y)$. Such an LFT can actually be found simply by a scaling and a translation, specifically $z \mapsto yz + x$.

Now, it’s easy to check that for this to be an isometry, it forces $ds^2 = \frac{dx^2 + dy^2}{y^2}$ (just take the derivative of that map).

We could do the same thing for the Poincare Disk model: look at the subgroup of Mobius transforms that fix the boundary of the disk. It turns out to be essentially the same as the above work because the disk is the same as the upper half-plane after applying an LFT like $z \mapsto i \left( \frac{z - i}{z + i} \right)$ that takes the upper half-plane to the disk. So the automorphisms of the disk model are conjugates of $\text{PSL}(2, \mathbb{R})$ by this particular element $z \mapsto i \left( \frac{z - i}{z + i} \right)$, or more explicitly, by things of the form $z \mapsto e^{i\theta} \left( \frac{z - a}{1 - \overline{a}z} \right)$.

Alternatively, we could get the metric on an arbitrary simply connected region in the plane by using the Riemann Mapping Theorem, which basically says that there is a unique complex analytic (conformal) map from the interior of any simple closed curve to the interior of the unit disk that takes a given point (with direction) to the center of the disk, with a certain direction.

**The Hyperboloid Model**

There are other models of hyperbolic geometry that don’t involve complex analysis. First we need the following fact, which seems like an exceptional coincidence from the theory of Lie groups.

**Proposition 1.** $\text{PSL}(2, \mathbb{R}) \simeq \text{SO}^+(2, 1)$

Here, $\text{SO}^+(2, 1)$ means we are talking about linear maps that preserve the quadratic form (or the cone) $x^2 + y^2 - z^2$. The $S$ means orientation-preserving as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$, and the + means it maps the upper cone to itself.

**Proof.** Some representations of $\text{SL}(2, \mathbb{R})$ are the defining representation (which acts on $\mathbb{R}^2$ by left multiplication like $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$), and another representation that acts on $\text{Sym}^2(\mathbb{R}^2)$ (2x2 symmetric matrices in 3-dimensional space), with the natural action on symmetric matrices being $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. More compactly, we can write this action as $M \circ S \rightarrow MSM^+$.

We want to show the image of this action is exactly $\text{SO}^+(2, 1)$. Since we’re in $\text{SL}(2, \mathbb{R})$ acting on $\text{Sym}^2(\mathbb{R}^2)$, it preserves the determinant: $\text{det} \begin{pmatrix} p & q \\ q & r \end{pmatrix} = pr - q^2$ is a quadratic form preserved by this action with signature $(1,2)$.

To figure out exactly which quadratic form this is, we can write this in the more standard form $\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$, which can be diagonalized to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. 

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Figure 3: The cone $x^2 + y^2 - z^2 = 0$

If we take $x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\det(x^y \\ z \\ x^{-y}) = x^2 - y^2 - z^2$.

(All of this is basically the same thing as looking at the action of $PSL(2, \mathbb{R})$ on its own Lie algebra.) Passing from $SL(2, \mathbb{R})$ to $PSL(2, \mathbb{R})$, we need to check the things in the kernel, and we multiply by -1 and -1, so we’re okay.

This leads to the Hyperboloid model of the hyperbolic plane: $\mathbb{H}^2 = \{ x^2 + y^2 - z^2 = -1, z > 0 \}$ (the top sheet of a hyperboloid with 2 sheets), with induced metric $ds^2 = dx^2 + dy^2 - dz^2$.

This is a good model for computations because you don’t need to divide by anything.

Figure 4: The hyperboloid $x^2 + y^2 = z^2 - 1$

Note that $dx^2 + dy^2 - dz^2$ on $\mathbb{R}^3$ is a Lorentzian metric, but it’s not obvious whether or
not it’s a Riemannian metric. Consider the hyperboloid \( x^2 + y^2 = z^2 - 1 \) with this metric. This is homogeneous and isotropic, and \( SO^+(2,1) \) acts transitively on this hyperboloid. To check the signature, it’s enough to check it at one conveniently chosen point, so we’ll check \((0,0,1)\). The tangent plane at this point is just the plane \( z = 1 \), and the induced metric is Riemannian.

Compare this model to the geometry of the sphere. We can do stereographic projection of the sphere onto the plane. By analogy, with this hyperboloid model, we can think of the hyperbolic plane as a sphere of radius \( i \). Many trig identities from spherical geometry can become hyperbolic identities this way.

Next time, we will talk about choosing the right model of the 3 models for your computations, and also we will discuss horocycles and Penner’s \( \lambda \)-length.