274 Curves on Surfaces: Lecture 10

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Last time we were looking at the cluster algebras from surfaces and various ways to construct them and we ended with the definition of a conjugated horocycle.

**Definition 1** (Conjugate horocycle). Let \( h \) be a horocycle around puncture, with length \( l(h) \). Then the conjugate horocycle is the one with length \( 1/l(h) \).

We ended with the Lemma 10.4. This lemma said that given a punctured monogon with some geometric structure, i.e. with some chosen horocycles around the punctures (in particular, we chose a horocycle and its conjugate) the product of the lambda-lengths of a geodesic measured with respect a horocycle by the lambda-length of the same geodesic with respect the conjugate horocycle, is equal to the lambda-length of an arc \( B \) looping around, i.e.

\[
\lambda(A)\lambda(A') = \lambda(B).
\]

An alternative way to write this is thinking about the distance between both horocycles. i.e. this distance is some reasonable function \( f(l(h)) \).

Let’s work out what this function is.

The hyperbolic distance between horocycles is

\[
l(A') - l(A) \quad (1)
\]

Exercise 10.5 tells us that

\[
\frac{\lambda(A')}{\lambda(A)} = l(h) \quad (2)
\]

So taking logarithms in 2 and using 1 we get

\[
l(A') - l(A) = 2\ln(l(h)).
\]

Let’s prove now Exercise 10.5

Proof of Q. Yuan.

It is clearly true when \( l(h) = 1 \) because in that case the horocycle is its own conjugate horocycle.

And in the general case we know that if we scale the horocycle by a factor of \( 1/c \) then the \( \lambda(A) \) scales by \( \sqrt{c} \), \( \lambda(A') \) scales by \( 1/\sqrt{c} \), and \( l(h) \) scales by a factor of \( c \).

Proof of D. Thurston.

We are looking at punctured monogons. Consider figure 1, What happens when \( l(h) = 1 \)? Then at height 1 the hyperbolic metric equals the euclidean, and the conclusion follows, \( l(h) = 1 \) as in the proof before.
For some height \( c \), the length of the horocycle is \( 1/c \).
For the conjugate horocycle the height is going to be \( 1/c \), then the length is \( c \). Then the distance between horocycles is

\[
\int_{\frac{1}{c}}^{c} \frac{1}{y} = 2 \ln(c)
\]

as we wanted to show.

Another exercise asked about the quiver \( \tilde{A}_{k,l} \), i.e. \( k \) arrows pointing clockwise and \( l \) arrows pointing counter clockwise.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{\( A_{k,l} \)}
\end{figure}

Aside: (background from cluster algebra): The direction of arrows does not matter in a tree. But it does if we have a loop.

Claim: \( \tilde{A}_{k,l} \) is mutation finite. Even more, it comes from some surface.

Let’s try to figure out which surface is starting by an hexagon with two punctures. We write down a triangulation
We draw the triangulation in red and freeze the middle edge.
Then we obtain $\tilde{A}_{6,2}$.

What does it mean to freeze the edge? It means that if we split the surface along that edge we get exactly the same thing. The have seen this in the case of a torus, were we froze the non identified edges, so we get

Finally we solve the last question. When we are looking at $D_4$, consider the punctured quadrilater with the following triangulation

were we have considered the quiver with clockwise arrows inside each triangle. The question is how can we go from the quiver shown in 5 to the last quiver shown in figure 6, which also gives the answer, namely, by mutating 1 and 2.
This is what we obtain from cluster algebra, but what does that mean in terms of the geometry?

The cluster variables correspond to edges in figure 5. Mutation of 1, by exchange relation means the product of $x_1$ by the mutated variable $x'_1$ is equal to the product of everything that points to plus the product of everything that points from, i.e.

$$x_1 x'_1 = x_4 + 1$$  \hspace{1cm} (3)

So geometrically we have to change $x_1$ to something else that satisfies 3. We claim that it should be the notch in figure 7 measured to the conjugate horocycle.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure7.png}
\caption{Mutation of 1 geometrically}
\end{figure}

We shall see why. Recall that according to figure 8

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure8.png}
\caption{}
\end{figure}

and looking the curve $y$ running around the puncture then, the exchange relation tells

$$y x_1 = x_3 x_4 + x_3 \cdot 1.$$  

So

$$y = x_3 [(x_4 + 1)/x_1]$$  \hspace{1cm} (4)

Thence using 3 in 4 we get
And by Lemma 10.4 we get that $x_1'$ is the length to the conjugate horocycle. Now we are also supposed to mutate edge 2. We get what figure 9 shows,

![Figure 9: Mutating 2 geometrically](image)

Now we mutate 4. Algebra tells us

$$x_4 x_4' = x_1 x_2 x_3 + 1 \tag{5}$$

As before, geometrically we have to replace $x_4$ for something satisfying equation 5, which is the pointed curve in figure 10,

![Figure 10: Mutating 4 geometrically.](image)

With the aim of clarity, let's pretend we have a loop $y$ around the puncture instead the pair of arcs $x_1$, $x_3$ as we show in figure 11.
The exchange relation tells that

\[ x_4x'_4 = 1 + x_2y. \]

Thence, now considering figure 10, we get the same result as in equation 5 since \( y = x_1x_3 \). So the geometric picture and the algebraic one match.

**Question from the audience:**
There is the triality relation in the \( D_4 \) quiver that seems to be totally hidden geometrically.

**Answer:** That’s right. It is some hidden symmetry of the considered Teichmüller space which is surprising from the geometric perspective and doesn’t show up in the geometry.

Basically surfaces give us all the classical theories of cluster algebras, but don’t give the exceptional ones. In particular, if we quotient \( D_4 \) by the triality we get \( G_2 \), which is exceptional and does not come from a surface.

Sometimes there are other cases where there are hidden symmetries from the geometrical perspective. For example the 4-punctured sphere. We triangulate it such that it looks like a tetrahedron and draw its quiver, as shows figure 12.
Then we pick the labels for the 6 different edges of the tetraedra and a clockwise circle on each face. If we flat figure 12 we get as the quiver the octaedron with oriented faces.

There is another way to decompose the same octaedron into triangles, by taking the dual tetrahedron. Here the hidden symmetry is a Regge symmetry. So far we’ve analyzed some examples, but let’s clarify the geometric meaning of what we’ve been doing.

**Definition 2.** A tagged simple arc on \((S,M)\) is an arc with one or both ends marked with a notch which does not self-intersect and does not bound a monogon or a 1-punctured monogon. Notches can only appear at punctures in the interior and should agree at common endpoints if an arc goes from a puncture to itself.

We want simple arcs, non self intersection and not to bound a monogon to rule out cases like

![Non-example of tagged arc](image13.png)

Figure 13: Non-example of tagged arc

Plus, we also want they not bounding a monogon, a 1-punctured monogon, to avoid

![Non-example of tagged arc](image14.png)

Figure 14: Non-example of tagged arc

Twice punctured is okay, though, as shows the triangulation

![Example of tagged arc (twice punctured is ok)](image15.png)

Figure 15: Example of tagged arc (twice punctured is ok)
Another special case to worry about is

![Figure 16: Non-example of tagged arc](image)

To rule out this case, we require in the definition that they should agree at common endpoints if an arc goes from a puncture to itself. The geometric meaning of the notch is that $\lambda$-lengths are measured respect to the conjugate horocycle. For example, if in the above example of $D_4$ we mutate the edge 3, we get two arcs both measured to the conjugate horocycle at that point, as shows figure 17.

![Figure 17: Mutation of 3 geometrically](image)

**Definition 3.** Two distinct tagged arcs are compatible if they don’t cross and if

- either tags agree at common endpoints
- or the arcs are parallel (isotopic to each other), one is notched and one is plain at one of the two endpoints.

Figure 18 shows cases examples that agree with the above definition of compatibility. Figure 19 shows non-examples of this compatibility notion.

![Figure 18: Examples of compatible tagged arcs](image)
Definition 4. A tagged triangulation fixing a punctured surface \((S, M)\) is a maximal collection of distinct compatible tagged arcs.

Theorem 1. Any tagged triangulation comes from ordinary triangulation by
(a) Replacing self-folded triangles by parallel arcs, as shows figure 20
(b) Flipping all the tags at some vertices, i.e., flipping some subset of the vertices where everything is notched, and changing everything being plane to everything being notched.

The idea of the proof is start with a collection of compatible arcs. For each vertex where you see only notched things you flip all those so that they become like the ordinary triangulation. If they are not all notched or all plane, then we are third case in figure 18, since there cannot be anything else attached to the vertex. Then we replace the notch.

The advantage of this result is that lets us use theorems that have been proved about collections of triangulations without having to reprove everything.

Once we have a tagged triangulation we can construct the quivers.
We recall the Lemma 10.4 that said that given two parallel arcs \(A, A'\), \((A'\) tagged), then
\[
\lambda(A)\lambda(A') = \lambda(B)
\]
, where \(B\) is the arc that loops around the two arcs. For instance, if we have some quadrilateral with ordinary triangulation, as in figure 21, then
where one of the sides is the boundary of some folded triangle. If we replace the self-folded triangle with parallel arcs, we double the corresponding vertex in the quiver, as shown in figure 22.

This shows how to construct the quiver from the triangulation. Conversely, we can go backwards, i.e. we can tell how a quiver comes from some triangulation using these rules. We deal with several types of basic building blocks.

The most simple type is what we get from a simple triangle with none of the sides bounding self-folded triangles (figure 23).

We can glue blocks to each other, but we do not allow gluing to itself at open point (not frozen vertices). We do allow cases shown in figure 24.
Where figure 24(a) shows another version of the $D_4$ surface. Note that we cancel opposite arrows after gluing.

We consider another basic building block, consisting of a simple arc (figure 25)

Figure 25: Type I

Figure 22 gives us another basic block, that we show in figure 26, with two frozen edges.

Figure 26: Type IV

Freezing one of the edges we get another type, shown in figure 27.

Figure 27: Type III

Finally we also have the final type shown in figure 28.
We could also have a triangle where two of the sides bound self-folded triangles as shown in figure 29.

![Figure 29: Triangle with two sides bounding self-folded triangles](image)

Its quiver comes from doubling two of the vertices of the triangle and connecting them in all possible ways (figure 30).

![Figure 30: Doubling vertices of triangle](image)

From the definitions it should be clear that there are only finitely many different distinct combinatorial types of tagged triangulations. So, everything we get from the above construction is mutation finite. But we don’t get everything which is mutation finite, in particular

**Exercise 1** (11.2). *It is not possible to obtain the exceptional series $E_6, E_7, E_8$ by gluing blocks.*

We state the following

**Theorem 2.** Every mutation finite symmetric cluster algebra is

1) rank 2

2) surface cluster algebra (gluing basic blocks)

3) exceptional series $E_6, E_7, E_8$, (figure 31)

4) $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ (adding extra vertex to the exceptional series diagram), (figure 32)
5) the elliptic algebras $\tilde{E}_6^{(1,1)}, \tilde{E}_7^{(1,1)}, \tilde{E}_8^{(1,1)}$, where we add some natural diagrams to the above. Since we are not longer dealing with trees, the orientation matter, so we orient them so that the triangles are cyclic. We show an example of the diagram added to $E_6$ in figure 33.

6) $X_6$ and $X_7$ in figure 34.
The current proof of this theorem involves computational techniques and dealing with several thousands of cases. It indeed deserved a more elegant proof which is posed as a challenging Problem.

To end with the lecture, we play around with a Java Applet that allows us to mutate quivers and we discuss a triangulation for an hexagon that gives $D_6$.

We also pose two more exercises

**Exercise 2.** What does figure 36 correspond in the list of mutate finite quivers? Can you mutate to get the standard form?
Exercise 3. Mutate figure 37

Figure 36: Exercise figure

Figure 37: Exercise figure