Zelevinsky's Talk

Questions?

Q: Is there a notion of complex length?

A: One amazing fact that started the whole notion of complex length:

Pick any surface, make a cut.

\[ \lambda \text{-length of this arc is a function of } \lambda \text{-lengths \ in } T \]

\[ \text{it is a rational func. because of each arc rel.} \]

\[ \text{Actually, it is a Laurent poly (by theory of Actors)} \]

Recall hyperbolic structures correspond to representations into \( \text{PSL}_2(\mathbb{R}) \).

(For closed surfaces)

\[ \text{For complex, } \lambda \text{-lengths can be interpreted as reps into } \text{PSL}_2(\mathbb{R}) = \text{iso} + (\mathbb{H}^3), \]

\[ \text{Unfortunately, } \lambda \text{-lengths can always be defined this way.} \]

\[ \lambda \text{-lengths not always well-defined} \]

Let's talk about these \( \lambda \)'s.
Closed surface with punctures

Isometries of $\mathbb{H}^2$:

**Elliptic**
- Fix a point in $\mathbb{H}^2$
- Rotate by some angle

**Hyperbolic**
- Fix a geodesic in $\mathbb{H}^2$
- Scale by a factor
- Fix two points on $\partial \mathbb{H}^2$
  (slide up?)

**In between:**

**Parabolic**
- Fix 1 pt at $\partial \mathbb{H}^2$

(Hyperbolic n-space isn't much worse than this, actually)
Whose does this classification come from?

**Proof:** Consider \( \text{det } M \in \text{SL}_2(\mathbb{R}) \). Can have 2 distinct real eigenvalues \( \lambda, \lambda^{-1} \) so it can be conjugated appropriately to get \( \begin{pmatrix} 
abla & 0 \\
0 & \lambda^{-1} \end{pmatrix} \)

\[ z \rightarrow z^\lambda, \text{ so } \lambda \text{ is hyperbolic} \]

Can have 2 complex equals \( \lambda \) so \( \lambda^{-1} = \bar{\lambda} \) on unit circle

\[ \begin{pmatrix} 
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \end{pmatrix} \rightarrow \text{rotation by } 2\theta \in \text{PSL}_2(\mathbb{R}) \]

\[ = \text{I} \in \text{PSL}_2(\mathbb{R}) \]

so elliptic.

Can have a repeated eigenvalue \( \lambda = 1 \), \( \lambda \begin{pmatrix} 1 & x \\
0 & 1 \end{pmatrix} \)

\[ z \rightarrow z + x, \text{ so parabolic} \]

\[ \text{decoration } \Rightarrow \text{choice of sign in } \mathbb{R}^3 \]

\[ \text{fixed by monodromy, } \text{modulo } \pm 1 (\text{or } 0 \text{ sign} ) \]
How do you get a B-matrix from a triangulation;

Put this little feature inside each $A,$ check this corresponds to changing $+2$ and $-2$.

That notation below is correct:

1. Add composite arrows through $v$
2. Reverse arrows to/from $v$
3. Delete bigons $\circledast$

Note: some vertices can be rotated, some can't:

Let:

get inner $G\sim A_3$ (note $0\rightarrow 0$)
and $0\rightarrow 0$
are mutation equivalent

Note arrows are clockwise in each $A$

Generally $A_n$ as hyperbolic structures on $(n+3)$-gon

assertation to see this is made and
you get by a line belt $T$
\( D_n \) = once punctured polygon

Or is mutation equivalent to

\[ \text{note same given at} \]

\[ \text{which is} \ A_\ell, \]

\( A_3, \)

so \( D_3 = A_3, \)

at root systems

punctured square \( D_4 \) = cluster algebra on \( \text{Mat}_{3 \times 3} \).
The mutation-finite cluster algebras fall into three classes:

1. Surface cluster algebras (i.e., $A_n$, $D_n$)
2. Rank 2 cluster algebras
3. $\mathbb{Z}/7$ diagrams (some with different short/long roots, e.g., $\mathbb{Z}/2$ and $\mathbb{Z}/2$ are not distinguished)

If we restrict to skew-symmetric (not stable), we get the simply-laced case: surfaces not orbifolds.

Given a surface $(S, M)$, look at set of anchors of $S$, these are in fact the arrows

\[ \text{e.g., } \text{plus} \quad \square \rightarrow \square \]

But certain modules $\text{MCG}$ is always finite, because if we don't use exactly when it is an surface, just unstructured data.

But which side of which a goes to which side of another a.

So translation $\text{MCG} \rightarrow \text{quiv}$

(note with $\alpha$-terms you get the quiver $\begin{array}{cc} 0 & 3 \\ \alpha & 0 \end{array}$)

So only finitely many quivers.

(alternatively, $\text{MCG}$ is only have entries $-1, 0, 1, 2, 3$ so)

There is only a finite # of possible $\text{MCG}$'s.
Exercise: Find surfaces that give the following series of Dynkin diagrams:

\[ A_k \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \]

(on-fa-sion; matter)

(same as previous)

(counterclock)

(very orientation)

From last time: \( x_{n+1} = x_n^2 + 1 \) (focus cut open with 2 big cylinders, annular)

\[ \begin{array}{c}
\text{Corner cases:} \\
\text{as}
\end{array} \]

\[ \lambda(E) \lambda(D) = \lambda(A) \lambda(C) + \frac{1}{\lambda(D)} \lambda(A) \]

\[ \begin{array}{c}
\text{vs.}
\end{array} \]

\[ x(D) x(D') = x(B) x(A) , \quad x(D') x(B) = x(A) x(C) , \quad x(D') x(D) = x(B) x(C) \]
**Def.** For \( p \) a cusp in a hyperbolic surface, \( h \) a horocycle around \( p \),

- let \( a = \) length of \( h \),
- let \( \overline{h} = \) conjugate horocycle
- horocycle of length \( \frac{1}{a} \)

**Lemma:** In a punctured monogon,

\[ \frac{A(A^1)}{A(A)} = \frac{1}{A(B)} \]

A and \( A^1 \) are same area, but in \( A^1 \), length is measured to conjugate horocycle

**Proof:**

- let \( a = \) length of horocycle
- recall

\[ \text{length of horocycle} = \frac{\lambda(\beta)}{\lambda(\alpha) \lambda(\gamma)} \]

so cutting obtuse, we get

\[ a = \frac{\lambda(\beta)}{\lambda(A^1)^2} \]

So for \( A^1 \), \( \frac{1}{a} = \frac{\lambda(\beta)^2}{\lambda(A^1)^2 \lambda(A)} \). Multiply these eqns, get \( 1 = \frac{\lambda(\beta)^2}{\lambda(A)^2 \lambda(A^1)^2} \). \( \Box \)