ADDITIVE CATEGORIFICATION OF (SURFACE) CLUSTER ALGEBRAS

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(Notes by Dylan Thurston)

We would really like a monoidal categorification, but it seems quite difficult and involves a lot of pain. Today we will see the additive categorification, which is still a little painful but is the best we have so far.

**General idea** Let \( C \) be a triangulated 2-Calabi-Yau \( \mathbb{C} \)-category with a cluster tilting object \( T = T_1 \oplus \cdots \oplus T_n \).

**Definition 1.** Triangulated: similar to an abelian category. Main feature: has self-equivalence \( \Sigma : C \to C \) and a collection of distinguished triangles

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X
\]

(replacement for the short exact sequences) which fulfill certain axioms. For example, for each morphism \( X \xrightarrow{f} Y \) there exists such a distinguished triangle containing \( f \). This is not unique, but also not completely arbitrary. For example, the derived category of an abelian category is a triangulated category.

**Definition 2.** 2-Calabi-Yau: There is a natural isomorphism \( \mathcal{C}(X,Y) \to \mathcal{D}\mathcal{C}(Y,\Sigma^2 X) \). Natural: functorial in both arguments. \( \mathcal{D} \): \( k \)-linear duality. (This is necessary to get the covariance to match.) The “2” refers to the 2 in \( \Sigma^2 X \). The name comes from algebraic geometry. In general, for a variety, there is a Serre duality \( \mathcal{C}(X,Y) \to \mathcal{D}\mathcal{C}(Y,\Sigma X) \) for some endofunctor \( S \). For a Calabi-Yau variety, \( S \) is just a shift.

In particular, \( \text{Ext}^1(X,Y) \cong \mathcal{C}(X,\Sigma Y) \cong \mathcal{D}\mathcal{C}(Y,\Sigma X) \cong \mathcal{D}\text{Ext}^1(Y,X) \). This is quite unusual.

**Definition 3.** Cluster tilting \( T \) means in this setup that for each \( X \in C \) indecomposable,

\[
\mathcal{C}(T,\Sigma X) = 0 \iff X \text{ direct summand of } T.
\]

This implies that have functor

\[
\mathcal{F} : \mathcal{C} \to \text{End}_C(T)^{\text{op}}\text{-Mod}
\]

\[
Z \mapsto \mathcal{C}(T,\Sigma Z)
\]

\( \ker(\mathcal{F}) \) is just given by morphisms which factor through \( T \).

For 2-Calabi-Yau, all the morphism spaces must be finite-dimensional. Also, the endomorphism algebra is finite-dimensional. Then see that

\[
\text{End}_C(T)^{\text{op}} = \mathbb{C}Q/I
\]

where \( Q \) is a quiver and \( I \) is an admissible ideal. (Admissible: quotient is finite-dimensional, don’t kill idempotents or arrows). This quiver and ideal is a basic invariant of finite-dimensional basic algebra.
Example 1. A typical example:

\[ \begin{align*}
1 & \overset{\alpha}{\rightarrow} 2 \overset{\beta}{\rightarrow} 3 \overset{\gamma}{\rightarrow} 1 \\
I & = \langle \beta \alpha, \gamma \beta, \alpha \gamma \rangle
\end{align*} \]

In many “good” situations, the cluster tilting objects behave just like clusters: For each \( k = 1, \ldots, n \), there exists a unique \( T_k^* \) so that \((T/T_k) \oplus T_k^*=: \mu_k(T)\) is again a cluster tilting object, and \( Q_{\mu_k(T)} = \mu_k(Q_T) \).

**Question 2.** What happens to the ideal \( I \)? **Answer:** in good situations, there is a rule for how \( I \) transforms as well. But it is awkwardly technical to state.

This is all very nice, but what does it have to do with cluster algebras? Detected by the cluster character (Caldero-Chapoton, Palu, Keller),

\[ C^T : \text{Obj}(C) \rightarrow \mathbb{Z} \left[ x_{\pm}^{\pm}, \ldots, x_{\pm}^{\pm} \right] \]

\[ Z \mapsto \mathcal{J} g(z) \sum_{\xi} \chi(\text{Gr}_Z^{op}(\mathcal{F}Z)) \hat{y}^\xi \]

\( T \): reference cluster tilting object

\[ g(Z) : \Sigma^{-1}Z \rightarrow T^b_Z \rightarrow T^a_Z \rightarrow Z \quad \text{distinguished triangle} \quad T^a_Z = T^a_1 \oplus \cdots \oplus T^a_n \]

\[ g(z) = a - b \in \mathbb{Z}^n \]

Use the Euler characteristic of the quiver Grassmannian: \( \text{Gr}_Z^Q(M) \) is the projective variety of all submodules of \( M \) which have dimension \( \underline{e} \). (Need to do some work to convert this to a projective variety.)

\( \hat{y} \) is a monomial change of basis:

\[ \hat{y}_k = \prod_{i=1}^{n} x_i^{\#(Q(i,k)) - \#(Q(k,i))} \]

\( Q(i,k) \): number of arrows from \( i \) to \( k \). If there are no 2-cycles, only one of these will be non-zero. (This is like the cross-ratios/shear coordinates in the surface algebra case.)

Ideally:

\[ C^T(Z) \in \mathcal{U}(Q, \underline{e}) \subset \mathbb{Z}[x_{1}^{\pm}, \ldots, x_{n}^{\pm}] \]

and indecomposable rigid objects (\( \mathcal{C}(Z, \Sigma Z) = 0 \)) give cluster variables.

This formula is somewhat complicated, but at least it’s a closed formula, at least if we can compute quiver Grassmannians.

**Example 3.** \( Q \) example from before. Consider the module

\[ 
\begin{array}{cccc}
\cdot e = (0, 0, 0) : & \text{pt} \\
\cdot e = (0, 0, 1) : & \text{pt} \\
\cdot e = (0, 1, 1) : & \mathbb{P}^1 \\
\end{array} 
\]

Reineke: quiver Grassmannians can be an arbitrary projective variety.
We can recognize the images of rigid objects under $F$ in $\text{End}_{C}^{\text{op}}(T)\text{-Mod}$: For $M \in (A = \text{End}_{C}(T))\text{-Mod}$, take a minimal projective presentation $P_1 \xrightarrow{\pi} P_0 \to M \to 0$ and then look at
\[
\text{Hom}_{A}(P_0, M) \xrightarrow{\text{Hom}(\pi, M)} \text{Hom}_{A}(P_0, M) \Rightarrow E(M) \to 0
\]
$M = F Z$ with $Z$ rigid $\iff E(M) = 0$.

(In particular, need $\text{Ext}^1_{A}(M, M) = 0$, but the condition is a little stronger.)

**Message:** Can forget about the triangulated category if you like and just work in the endomorphism algebra, which is more elementary, although less symmetric.

How can we find such categories $C$ with cluster tilting object $T$?

Answer: Quivers with potential (non-degenerate).

Potential $W$ is a linear combination of cycles in $Q$.

**Example 4.** $Q =$ Markov quiver, arrows $a_0, a_1, \ldots, c_0, c_1$. $W = c_0b_0a_0+c_1b_1a_1+\lambda c_0b_1a_0c_1b_0a_1$. (This is a typical potential.)

From this data, can construct (Amit, Keller) a Ginzburg dg-algebra, then a 2-Calabi-Yau category $C$ with cluster tilting object $T$, and $\text{End}_{C}(T)^{\text{op}} = CQ/\langle \partial W \rangle$, the Jacobian algebra.

Here, $\partial W$ is the cyclic partial derivative. Eg, in the example above,
\[
\partial_{a_0} W = c_0b_0 + \lambda c_1b_0a_0c_0b_1.
\]

Get six generators of the Jacobian ideal in this way.

(Really need to take some completions to be precise. In this example, if you take the path algebra modulo the six relations, get an infinite-dimensional algebra. The better thing is the completed path algebra modulo the closure of the relations.)

**Exercise 1.** (Hard) For the Markov quiver example, the completed path algebra $\hat{C}Q/\langle \partial W \rangle$ is finite dimensional, while $CQ/\langle \partial W \rangle$ is not finite-dimensional.

(Part of exercise: look up definitions in Derksen-Weyman-Zelevinsky’s paper.)

**Exercise 2.** (Easy) For
\[
Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 1
\]
$W = cba$

then $CQ/\langle \partial W \rangle = \hat{C}Q/\langle \partial W \rangle$ and it is just
\[
\partial W = \{ba, cb, ac\}
\]
as in the earlier example.