Lecture 21: Linear maps

November 30, 2010

Announcements

  ► Leave it in drop box next to Math 406.
  ► Please try to do the exercises before Thursday’s lecture.

► Get a copy of handout: “Linear maps, the total derivative and the chain rule”.

► On the second midterm, I graded some students too harshly on the complex multiplication question. If you left your answer for 2(a) and 2(b) in a form with cos and sin, please hand it in for regrading.
Lecture 21: Linear maps

Functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$

Composition of functions

Linear maps

Matrices
Functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$

In Calculus I and II you study
▶ Functions $\mathbb{R} \rightarrow \mathbb{R}$
So far in this course we have also seen:
▶ Parametric curves: Functions $\mathbb{R} \rightarrow \mathbb{R}^n$
  (Studied: Velocity, speed, acceleration)
▶ Functions of many variables: Functions $\mathbb{R}^n \rightarrow \mathbb{R}$
  (Studied: Partial derivatives, critical points, etc.)
We will now turn to the general situation:
▶ Functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$
These are functions that take as input a vector in $\mathbb{R}^n$
and output a vector in $\mathbb{R}^m$.
There are many such functions! We’ll start by building up some examples to work with.

Examples
▶ $\mathbb{R} \rightarrow \mathbb{R}$: $f(x) = e^x + x^2$
▶ $\mathbb{R} \rightarrow \mathbb{R}^3$:
  $\vec{r}(t) = (\sin(t), \cos(t), 1)$
▶ $\mathbb{R}^2 \rightarrow \mathbb{R}$:
  $f(x, y) = x^2 - y^4 + xy$
▶ $\mathbb{R}^2 \rightarrow \mathbb{R}^2$:
  Rotation by 90° counterclockwise:
  $f(x, y) = (-y, x)$
▶ $\mathbb{R}^3 \rightarrow \mathbb{R}^2$:
  Projection of a vector in $\mathbb{R}^3$ onto xy-plane
  $f(x, y, z) = (x, y)$.
Examples, $\mathbb{R}^2 \to \mathbb{R}^2$

We'll start with functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

- Rotation by $90^\circ$ counterclockwise:
  $f(x, y) = (-y, x)$.
- Rotation by $90^\circ$ clockwise:
  $f(x, y) = (y, -x)$.
- Translation by a the vector $\langle 1, 2 \rangle$:
  $f(x, y) = (x + 1, y + 2)$.
- Conversion from polar coordinates to Cartesian coordinates:
  $f(r, \theta) = (r \cos \theta, r \sin \theta)$.
- The map $f(x, y) = (x, x + y)$ is a shear.
- The map $f(x, y) = (2x, 2y)$ scales by a factor of 2.

- The map that takes a complex number $z$ to $z^2$ can be thought of as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$:
  
  $f(x, y) = (x + iy)^2$
  
  $= (x^2 + 2ixy + (iy)^2)$
  
  $= (x^2 - y^2) + i(2xy)$
  
  $= (x^2 - y^2, 2xy)$

- Rotation by a general angle $\theta$:
  
  $f(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$.

  Can also think of this as multiplication of a complex number by $e^{i\theta}$:
  
  $f(x, y) = (x + iy)(e^{i\theta})$
  
  $= (x + iy)(\cos \theta + i \sin \theta)$
  
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Examples, $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

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- **The map $f(x, y) = (x, x + y)$ is a shear.**

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  f(x, y) = (x + iy)(e^{i\theta}) = (x + iy)(\cos \theta + i \sin \theta) = (x \cos \theta - y \sin \theta) + i(x \sin \theta + y \cos \theta) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)
  \]
Examples, $\mathbb{R}^n \to \mathbb{R}^m$

- The function $P$ that takes a vector in $\mathbb{R}^3$ to its projection on the xy-plane is a function $\mathbb{R}^3 \to \mathbb{R}^2$.
- Rotation around z-axis by an angle of $\pi/4$ is a function $\mathbb{R}^3 \to \mathbb{R}^3$.
- The function that converts from cylindrical to Cartesian coordinates is a function $\mathbb{R}^3 \to \mathbb{R}^3$: 
  $$F(r, \theta, z) = (r \cos \theta, r \sin \theta, z).$$
- For $f(x, y)$ a function of two variables (e.g., $f(x, y) = x^2 - y^2$), 
  $$F(x, y) = (x, y, x^2 - y^2)$$
  is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Note that a function $\mathbb{R}^n \to \mathbb{R}^m$ can be conveniently split up into $m$ different functions $\mathbb{R}^n \to \mathbb{R}$, the same way we split up a function $\mathbb{R} \to \mathbb{R}^3$ into three different functions $\mathbb{R} \to \mathbb{R}$: 
  $$\vec{r}(t) = (x(t), y(t), z(t))$$
Lecture 21: Linear maps

Functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$

Composition of functions

Linear maps

Matrices
One useful operation with functions is composition. Given a function \( F : \mathbb{R}^n \to \mathbb{R}^m \), and a function \( G : \mathbb{R}^m \to \mathbb{R}^p \), can form their composite \( G \circ F \):

\[
(G \circ F)(\vec{v}) = G(F(\vec{v})).
\]

In terms of coordinates, \( G \circ F \) means substituting variables. \( G \circ F \) only makes sense if the dimensions match up. Composition is not commutative: it is not always true that \( G \circ F = F \circ G \).

**Examples**

Let \( T \) be translation in \( \mathbb{R}^2 \) by \( \langle 1, 2 \rangle \) and let \( R_\theta \) be rotation in \( \mathbb{R}^2 \) by angle \( \theta \).

- \( T \circ T \) is translation by \( \langle 2, 4 \rangle \).
- \( R_{\pi/2} \circ R_{\pi/2} \) is the function \( (x, y) \mapsto (-x, -y) \), rotation by \( \pi \).
- What is \( R_{\pi/2} \circ R_{\pi/4} \)?

Let \( P \) be the projection from \( \mathbb{R}^3 \) onto the xy-plane, \( P(x, y, z) = (x, y) \).

- \( R_{\pi/4} \circ P \) is projection followed by rotation, but \( P \circ R_{\pi/4} \) doesn't make sense.
- What is \( R_{\pi/2} \circ T \)? What is \( T \circ R_{\pi/2} \)?
Use some basic building blocks:

- $P$ is the projection from $\mathbb{R}^3$ onto the $xy$-plane, $P(x, y, z) = (x, y)$.
- $R^z_\theta$ is the rotation of $\mathbb{R}^3$ by $\theta$ around the $z$ axis.
- $R_\theta$ is the rotation of $\mathbb{R}^2$ by $\theta$ around 0.

Exercise
- What is $R^z_{\pi/4}$ in coordinates?
- What is $P \circ R^z_{\pi/4}$?
- What about $R^z_{\pi/4} \circ P$?
- Show that $P \circ R^z_{\pi/4} = R_{\pi/4} \circ P$, where $R_{\pi/4}$ is rotation in $\mathbb{R}^2$. 
Lecture 21: Linear maps

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Linear maps: Definition

Definition
A linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is one that can be written
$$f(x, y) = (ax + by, cx + dy)$$
for some real numbers $a, b, c, d$.

More generally, a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is one that can be written
$$f(x_1, \ldots, x_n) =$$
\begin{align*}
(a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n, \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n, \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n).
\end{align*}

Examples
- $f(x, y) = (x + y, y)$ (shear) is linear.
- $f(x, y) = (2x, 2y)$ (scaling) is linear.
- $R_{\pi/2}$ is linear.
- In general, $R_\theta$ is linear:
  $$R_\theta(x, y) = (x \cos \theta - y \sin \theta, \ \
x \sin \theta + y \cos \theta)$$
- Complex squaring map does not seem linear:
  $$f(x, y) = (x^2 - y^2, 2xy)$$
- Translation $T$ does not seem linear.
- The 3D rotation $R^{\pi/4}_{\pi/4}$ is linear.
- The projection $P(x, y, z) = (x, y)$ is linear.
- $f(x, y) = x^2 + y^2$ does not seem linear.
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- \( f(x, y) = x^2 + y^2 \) does not seem linear.
Linear maps: Composition

Lemma

The composition of two linear maps is a linear map. That is, if $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^p$ are both linear, then $G \circ F : \mathbb{R}^n \to \mathbb{R}^p$ is also linear.

Proof.

We'll only do case of maps $\mathbb{R}^2 \to \mathbb{R}^2$. General case is similar, but notation more annoying. Have

$$F(x, y) = (ax + by, cx + dy)$$
$$G(x, y) = (ex + fy, gx + hy).$$

Then

$$(G \circ F)(x, y) = G(ax + by, cx + dy)$$
$$= (e(ax + by) + f(cx + dy),$$
$$g(ax + by) + h(cx + dy))$$
$$= ((ae + cf)x + (be + df)y,$$
$$ (ag + ch)x + (bg + dh)y). \qed$$

Example

Consider the composition $R_{\pi/2} \circ R_{\pi/4}$. 
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$g(ax + by) + h(cx + dy))$

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Linear maps: Composition

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The composition of two linear maps is a linear map. That is, if $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^p$ are both linear, then $G \circ F : \mathbb{R}^n \to \mathbb{R}^p$ is also linear.

Proof.

We'll only do case of maps $\mathbb{R}^2 \to \mathbb{R}^2$. General case is similar, but notation more annoying. Have

$F(x, y) = (ax + by, cx + dy)$

$G(x, y) = (ex + fy, gx + hy)$.

Then

$\begin{align*}
(G \circ F)(x, y) &= G(ax + by, cx + dy) \\
&= (e(ax + by) + f(cx + dy), \\
&\quad g(ax + by) + h(cx + dy)) \\
&= ((ae + cf)x + (be + df)y, \\
&\quad (ag + ch)x + (bg + dh)y).
\end{align*}$
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$$(G \circ F)(x, y) = G(ax + by, cx + dy)$$
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Example

Consider the composition $R_{\pi/2} \circ R_{\pi/4}$. 
Linear maps: Properties

Lemma
A linear map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ preserves vector addition and scalar multiplication. That is, if $\vec{v}, \vec{w} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

\[
F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})
\]

\[
F(\lambda \vec{v}) = \lambda F(\vec{v})
\]

This lemma is often taken as a definition of linear maps.

Proof.
Will just prove 2D case, and only for addition. Set

$F(x, y) = (ax + by, cx + dy)$

$\vec{v} = (r, s)$  $\vec{w} = (t, u)$.

Then

\[
F(\vec{v} + \vec{w}) = F(r + t, s + u)
\]

\[
= (a(r + t) + b(s + u), c(r + t) + d(s + u))
\]

\[
= (ar + bs, cr + ds) + (at + bu, ct + du)
\]

\[
= F(\vec{v}) + F(\vec{w}).
\]

Example
We can use this to show that functions are not linear.

- Translation $T$ is indeed not linear.
- The complex squaring map is indeed not linear.
- The function $f(x, y) = x^2 + y^2$ is indeed not linear.
- The function $f(x) = |x|$ is not linear.
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\[
F(x, y) = (ax + by, cx + dy)
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= (a(r + t) + b(s + u), c(r + t) + d(s + u))
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We can use this to show that functions are not linear.
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Lecture 21: Linear maps

Functions $\mathbb{R}^n \to \mathbb{R}^m$

Composition of functions

Linear maps

► Matrices
Consider again representation of a linear map \( \mathbb{R}^2 \to \mathbb{R}^2 \):

\[
F(x, y) = (ax + by, cx + dy).
\]

It is convenient to put \( a, b, c, d \) in a rectangular array:

\[
[F] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

This array is the matrix of \( F \). It is (here) a shorthand for a linear map.

**Examples**

- \( R_{\pi/2}(x, y) = (-y, x) \) is represented by
  \[
  [R_{\pi/2}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
  \]

- \( R_\theta \) has matrix
  \[
  [R_\theta] = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
  \]

- What is the matrix of the shear function \( f(x, y) = (x + y, y) \)?

- What is the matrix of the do-nothing (identity) function \( f(x, y) = (x, y) \)?
Matrices in general

For a general linear map

\[ F(x_1, \ldots, x_n) = (a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n, \]
\[ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n, \]
\[ \cdots, \]
\[ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n), \]

its matrix is

\[ [F] = \begin{pmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{pmatrix} \]

▶ The matrix of \( R_{\pi/4}^z \) is

\[ [R_{\pi/4}^z] = \begin{pmatrix}
  \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\
  \sqrt{2}/2 & \sqrt{2}/2 & 0 \\
  0 & 0 & 1
\end{pmatrix}. \]

▶ What is the matrix of reflection in the \( xy \)-plane?

▶ What is the matrix of projection \( P \) from \( \mathbb{R}^3 \) onto the \( xy \)-plane?
Of course, we also want to be able to go backwards: Given the matrix \([F]\), find how to compute \(F(\vec{v})\).

- The \(i^{th}\) entry of \(F(\vec{v})\) is the dot product of the \(i^{th}\) row of \([F]\) with \(\vec{v}\).

Another technique (which will come up again):
- Write \(\vec{v}\) as a column vector.
- The \(i^{th}\) entry of \(F(\vec{v})\) is obtained by
  - running your left index finger along the rows of \([F]\) and your right index down \(\vec{v}\),
  - multiplying the entries as you go, and
  - adding up the result.

**Example**

Suppose \(F(x, y) = (x + 2y, 3x + 4y)\), so
\[
[F] = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.
\]

Then \(F(7, 8)\) is
\[
F(7, 8) = ((1, 2) \cdot (7, 8), (3, 4) \cdot (7, 8))
= (7 + 16, 21 + 32)
= (23, 53).
\]

Alternatively, write \((7, 8)\) as the column vector \((7) \begin{pmatrix} 8 \end{pmatrix}\) and compute
\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix}
\]
by running left index finger across rows and right index finger down column of \((7) \begin{pmatrix} 8 \end{pmatrix}\).