Lecture 20: Lagrange multipliers, II

November 23, 2010

Announcements

► No office hours on Wednesday
► Next office hours: Monday, November 29, 2–3PM, Math 614
► Problem set 10 will be due Thursday, December 2.
  ▶ Will include some material from next Tuesday’s lecture.
► Upcoming material on linearization not in textbook
  ▶ Covered in a handout originally written by Prof. Robert Lipshitz.
  ▶ Available online. Printed copy will be distributed on Tuesday. (Check for updates.)
Lecture 20: Lagrange multipliers, II

▶ Reprise: Gradients &c

3-dimensional case

Two constraints

Pretty pictures
Story so far
For a function \( f(\cdot, \cdot) \), optimizing in bulk (with no constraints):

- Gradient \( \nabla f \) points in dir. of steepest increase of \( f \).
- \( \nabla f \) is perp. to contour lines.
- Local max/min are some of critical points, where \( \nabla f = 0 \).

If there is a constraint \( g(x, y) = k \):

- No longer allowed to move in every direction. Gradient must be perpendicular to constraint curve.
- Boils down to \( \nabla f = \lambda \nabla g \). (E.g., \( \nabla f = 0 \) is still OK.)

With no constraints, there is also second deriv. test, checking \( f_{xx}f_{yy} - (f_{xy})^2 \). Could also check this with constraints, but usually not worth it.

Example
What is the maximum of \( f(x, y) = xy \) on the circle \( x^2 + y^2 = 1 \)?

- \( \nabla f = \langle y, x \rangle \).
- \( \nabla f = 0 \) only at \((x, y) = (0, 0)\). This is a saddle point.

On boundary, need not have \( \nabla f = 0 \) at local max. Only need \( \nabla f \) perp. to boundary.

- Take \( g(x, y) = x^2 + y^2 \).
- \( \nabla g = \langle 2x, 2y \rangle \).
- Solve

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x^2 + y^2 = 1 \\
\langle y, x \rangle = \lambda \langle 2x, 2y \rangle.
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Reprise: Gradients &c

3-dimensional case

Two constraints

Pretty pictures
Lagrange multipliers in 3 dimensions

What happens in 3 dimensions?
Suppose we want to optimize $f(x, y, z)$ subject to constraint $g(x, y, z) = K$.

Suppose $g(x, y, z) = K$ defines a smooth surface. $\nabla f(a, b, c)$ must be perpendicular to this surface at optimum point $(a, b, c)$.

Surface is contour of $g(\cdot, \cdot, \cdot)$. So $\nabla g(a, b, c)$ is also perp. to surface. Therefore $\nabla f(a, b, c)$ and $\nabla g(a, b, c)$ are parallel.

Solve:

$g(x, y, z) = K$

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

($\lambda$ is unknown, the Lagrange multiplier.)

Example
Find point on sphere $x^2 + y^2 + z^2 = 1$ closest to $(1, 2, 3)$.
This boils down to: minimize $f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$
subject to constraint $x^2 + y^2 + z^2 = 1$,
i.e., $g(x, y, z) = 1$ where $g(x, y, z) = x^2 + y^2 + z^2$. 
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Example: Lemonade, II

Problem
A lemonade stand is making lemonade, iced tea with lemon, and mini lemon cakes. Their profit as a function of glasses $l$ of lemonade, $t$ of tea, and $c$ cakes (in pennies) is

$$p(l, t, c) = 100l - l^2 + 50t - t^2 + 240c.$$  

However, they only have 120 lemons, and it takes 6 lemons per slice of cake, 3 per glass of lemonade, and 1 per cup of tea. Maximize profit.

Solution, II

As we said, the best solution will use all the lemons:

$$3l + t + 6c = 120$$

Instead of solving for $c$ and substituting like before, treat this as Lagrange multipliers to find critical points. With $g(l, t, c) = 3l + t + 6c$, solve:

$$\nabla g = \lambda \nabla p$$

$$\langle 3, 1, 6 \rangle = \lambda \langle 100 - 2l, 50 - 2t, 240 \rangle$$

Still not the right solution!
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3-dimensional case

Two constraints

Pretty pictures
Two constraints: The problem

Sometimes we'll want to optimize a function \( f(\cdot, \cdot, \cdot) \) on a curve in space.

If the curve is given parametrically, by a function \( \vec{r}(t) \), can use the chain rule: Compute \( f(\vec{r}(t)) \).

Function of one variable.

If curve is given by intersection of surfaces, need to work more.

More constraints also come up often; techniques are similar.

Example

- Maximize \( x + y + z \) on the curve \( \vec{r}(t) = (\cos(t), \sin(2t), 1) \).
- Maximize \( x + y + z \) when
  \[
  x^2 + y^2 + z^2 = 1 \\
  x^2 - y^2 + z = 0
  \]
Intersection of two surfaces is (usually) a curve.
The tangent vector to the intersection is contained in tangent planes to both surfaces.
Tangent vector is perpendicular to both gradient vectors. Given by cross product.

Example
Maximize \( x + y + z \) when
\[
\begin{align*}
x^2 + y^2 + z^2 &= 1 \\
x^2 - y^2 + z &= 0
\end{align*}
\]
Lagrange multipliers

Now maximize $f(\cdot, \cdot, \cdot)$ on curve given by intersecting surfaces:

\[ g(x, y, z) = k \]
\[ h(x, y, z) = l \]

Let $\vec{v}$ be tangent vector to curve of intersection at some point. $\vec{v}$ is perp. to $\nabla g, \nabla h$:

\[ \vec{v} \cdot \nabla g = 0 \]
\[ \vec{v} \cdot \nabla h = 0 \]

Any combination of these two (e.g., $\nabla g + 2\nabla h$) will also be perp. to $\vec{v}$.

At a maximum, $\vec{v}$ must also be perp. to $\nabla f$. This happens if

\[ \nabla f = \lambda \nabla g + \mu \nabla h \]

for unknown $\lambda, \mu$.

Example

Maximize $x + y + z$ when

\[ x^2 + y^2 + z^2 = 1 \]
\[ x^2 - y^2 + z = 0 \]

Method

Set

\[ f(x, y, z) = x + y + z \]
\[ g(x, y, z) = x^2 + y^2 + z^2 \]
\[ h(x, y, z) = x^2 - y^2 + z. \]

Solve

\[ x^2 + y^2 + z^2 = 1 \]
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(100 - 2l, 50 - 2t, 240) = \lambda \langle 3, 1, 6 \rangle + \mu \langle 1, 0, 0 \rangle
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Of course, there are easier ways to do this particular problem...
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$$l = 0$$

With $g(l, t, c) = 6c + 3l + t$ and $h(l, t, c) = l$, solve:

$$3l + t + 6c = 120$$  
$$l = 0$$

$$\langle 100 - 2l, 50 - 2t, 240 \rangle = \lambda \langle 3, 1, 6 \rangle + \mu \langle 1, 0, 0 \rangle$$

Of course, there are easier ways to do this particular problem...
The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse. Find the lowest and highest points on the ellipse.

(Stewart, Exercise 14.8.42)
Reprise: Gradients &c

3-dimensional case

Two constraints

Pretty pictures
There are some really pretty constraint surfaces.

- \( z^2 + (x^2 + y^2 - 1)^2 = \frac{1}{2} \)
- \( z^2 + (x^2 + y^2 - 1)^2 = 1 \)
- \( (xy)^2 + (yz)^2 + (xz)^2 - xyz = 0 \)
- \( \cos(x) + \cos(y) + \cos(z) = 0 \)

Need to be careful when the gradient of function defining constraint surface is zero (\( \nabla g = 0 \)):
surface may have singularities.
Second derivative test

Put your pencils down; this is just for fun, a taste of what can come.
Here's the second derivative test for Lagrange multiplier problems:

**Theorem**
Suppose \( g(\vec{p}) = k, \nabla g(\vec{p}) \neq 0 \) and \( \nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) \). Let \( S = \{(x, y, z) \mid g(x, y, z) = k\} \). Let \( \vec{r} = (x(s, t), y(s, t), z(s, t)) \) be parametric equations for \( S \) near \( \vec{p} \). Then:

- If \( D > 0 \) and \( f_{ss} > 0 \) then \( \vec{p} \) is a local min of \( f \) on \( g = k \).
- If \( D > 0 \) and \( f_{ss} < 0 \) then \( \vec{p} \) is a local max of \( f \) on \( g = k \).
- If \( D < 0 \) then \( \vec{p} \) is a saddle.
- If \( D = 0 \) then the second derivative test failed.

**Examples**

- Height function on \( S^2 \).
- Function \( x \) on first example from last slide \( z^2 + (x^2 + y^2 - 1)^2 = 1/2 \) (a torus).
Morse theory

Theorem

Suppose the constraint surface $S = \{g = k\}$ is a sphere and $f$ a function which doesn’t fail the second derivative test (at any point with $\nabla f = \lambda \nabla g$). Let

$m = \text{number of local mins}$

$s = \text{number of saddles}$

$M = \text{number of local maxes}$.

Then $m \geq 1$, $M \geq 1$ and $m - s + M = 2$.

If $S$ is instead a torus then $m - s + M = 0$.

If $S$ is a surface of genus $g$ then $m - s + M = 2 - 2g$.

This is subject of topology. This kind of count lets us tell a donut from a beach ball (but not from a coffee cup)!