Linear approximations

Key point from last time: Any nice 2-variable function has a linear approximation from partial derivatives:

**Theorem**

If \( f(x, y) \) is differentiable at \((x_0, y_0)\), then

\[
f(x_0 + a, y_0 + b) \approx f(x_0, y_0) + af_x(x_0, y_0) + bf_y(x_0, y_0).
\]

For instance, this applies if \( f_x \) and \( f_y \) are continuous at \((x_0, y_0)\).
Use this to get the tangent plane:

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

where \( z_0 = f(x_0, y_0) \).

Chain rule: 1 → 2 → 1

Now suppose we have \( f(x(t), y(t)) \).
(Maybe \( f \) is the altitude, and \( x(t) \) and \( y(t) \) are my position as I move around.)

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]

Sum over all paths from \( t \) to \( f \).
Lecture 16: Directional derivatives and gradients

Reprise

► Directional derivatives

Gradient

Normals and tangents

Summary: Types of derivatives

Directional derivative: Intro

Partial derivatives tell us what slope is as we approach along x- or y-axis (east or north):

\[
 f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}
\]

\[
 f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}
\]

What happens if we approach from, say, southwest?

Point at distance h from \((x_0, y_0)\) to southwest is

\((x_0 + h/\sqrt{2}, y_0 - h/\sqrt{2}) = (x_0, y_0) + h\vec{u}\) where \(\vec{u} = (1/\sqrt{2}, -1/\sqrt{2})\) is unit vector pointing southwest.

Definition

If \(f(x, y)\) is a function and \(\vec{u} = (a, b)\) is a unit vector, the directional derivative of \(f\) in direction \(\vec{u}\) is

\[
 D_{\vec{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}
\]

(if limit exists).

Examples

- \(D_{(1,0)}f = f_x.\) \(D_{(0,1)}f = f_y.\)

Finding directional derivatives

Directional derivatives can behave badly in general. For differentiable functions, have

\[
 f(x_0 + a, y_0 + b) \approx f(x_0, y_0) + af_x(x_0, y_0) + bf_y(x_0, y_0)
\]

\[
 D_{\vec{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}
\]

\[
 = af_x(x_0, y_0) + bf_y(x_0, y_0).
\]

Theorem

If \(f\) is differentiable at \((x_0, y_0)\) and \(\vec{u} = (a, b)\) is a unit vector, then

\[
 D_{\vec{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0).
\]

Proof.

Use the chain rule: \(x(h) = x_0 + ah, y(h) = y_0 + bh.\) \(\square\)
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Directional derivatives

◮ Gradient

Normals and tangents

Summary: Types of derivatives

Gradient: Intro

The directional derivative we found looks like a dot product:
\[ D_{(a,b)} f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0) = (a, b) \cdot (f_x(x_0, y_0), f_y(x_0, y_0)). \]

Definition
The gradient of a function \( f(x, y) \) that is differentiable at \((x_0, y_0)\) is
\[ \nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0)). \]

We then have
\[ D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot \nabla f(x_0, y_0). \]

Steepest ascent: Meaning of gradient

We can use the basic formula
\[ D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot \nabla f(x_0, y_0). \]

to give a geometric meaning to the gradient.

Intuitive statement
The gradient points in the direction of steepest ascent, and its length is the slope.

Theorem
As \( \vec{u} \) varies, the directional derivative \( D_{\vec{u}} f(x_0, y_0) \) is maximized when \( \vec{u} \) points in the same direction as \( \nabla f(x_0, y_0) \). The maximal value is \( ||\nabla f(x_0, y_0)|| \).
\[ D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot (\nabla f(x_0, y_0)) \]

Proof.
\[ = ||\vec{u}|| \cdot ||\nabla f(x_0, y_0)|| \cdot \cos \theta \]
\[ = ||\nabla f(x_0, y_0)|| \cdot \cos \theta. \]
This is maximized when \( \cos \theta = 1 \), i.e., \( \theta = 0 \).
Lecture 16: Directional derivatives and gradients

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Directional derivatives

Gradient

Normals and tangents

Summary: Types of derivatives

Contour lines and gradient

Theorem
For a differentiable function \( f(x, y) \), the gradient vector at \((x_0, y_0)\) is perpendicular to the contour line through \((x_0, y_0)\).

Proof.
The contour line is determined by \( f(x, y) = c \) for constant \( c \).
Parametrize the contour line by \( x(t), y(t) \), with \( x(0) = x_0 \) and \( y(0) = y_0 \).
Tangent to contour line is \( (\frac{dx}{dt}, \frac{dy}{dt}) \).
\[
0 = \frac{d}{dt} f(x(t), y(t)) \\
= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\
= \nabla f \cdot \vec{v}
\]
(Points of evaluation are omitted for brevity.)

3-dimensional case
Suppose we have a function of three variables: \( f(x, y, z) \). What can we do then?

Contour plot is a collection of surfaces in space.
Have three partial derivatives: \( f_x, f_y, f_z \).
Directional derivative as before: for \( f \) differentiable, \( \vec{u} = (a, b, c) \) a unit vector,
\[
D_\vec{u} f(x_0, y_0, z_0) = af_x(x_0, y_0, z_0) + bf_y(x_0, y_0, z_0) + cf_z(x_0, y_0, z_0) \\
= \vec{u} \cdot \nabla f(x_0, y_0, z_0).
\]

Gradient vector \( \nabla f = (f_x, f_y, f_z) \).

Theorem
For a differentiable function \( f(x, y, z) \), the gradient vector at \((x_0, y_0, z_0)\) is perpendicular to the contour surface through \((x_0, y_0, z_0)\).

Tangent plane to contour surface at \((x_0, y_0, z_0)\) has normal \( \nabla f(x_0, y_0, z_0) \):
\[
\{(x, y, z) \mid f_x(x_0, y_0, z_0)(x-x_0) + f_y(x_0, y_0, z_0)(y-y_0) + f_z(x_0, y_0, z_0)(z-z_0) = 0\}
\]

Question
What about the case \( f(x, y, z) = ax + by + cz + d? \quad f(x, y, z) = z - g(x, y)? \)
Summary

What might we mean by “derivative” of a function like \( f(x, y) = x^3 + y^3 + xy \) at \((1, 2)\)?

- **Partial derivative**
  \[
  \frac{\partial f}{\partial x}(1, 2) = f_x(1, 2) = 5 \quad \frac{\partial f}{\partial y}(1, 2) = f_y(1, 2) = 7
  \]

- **Linear approximation**
  \[
  \begin{align*}
  f(1 + a, 2 + b) &\approx f(1, 2) + 5a + 7b \\
  f(x, y) &\approx f(1, 2) + 5(x - 1) + 7(y - 2) \\
  \Delta f &\approx 5\Delta x + 7\Delta y
  \end{align*}
  \]

- **Tangent plane**: \((z - f(1, 2)) = 5(x - 1) + 7(y - 2)\). Normal: \((1, 5, 7)\).

- **Directional derivative** in direction of \(\vec{v} = (1, 1)\):
  \[
  D_{(1/\sqrt{2}, 1/\sqrt{2})}f(1, 2) = 5(1/\sqrt{2}) + 7(1/\sqrt{2}) = 12/\sqrt{2} = 6\sqrt{2}
  \]

- **Gradient**:
  \[
  \nabla f(1, 2) = (5, 7)
  \]