Announcements

▶ School break Monday, November 1 and Tuesday, November 2. Go vote!
▶ Homework 7 due Thursday, November 4.
▶ Next office hours: Wednesday, 2:00–3:00, Math 614.
Derivatives of functions of two variables

Today we’ll learn what a “derivative” of a function of two variables could be.

For ordinary functions, the derivative of $f(x)$ encodes the amount that $f(x)$ changes if the argument $x$ changes by a small amount.

For functions of two variables $f(x, y)$, the arguments can change in lots of different ways! How to keep track of it all?

We’ll start off today by only looking at the change parallel to one of the axes.
Lecture 14: Partial derivatives

- Review: Derivatives in 1D

  Partial derivatives

  Higher derivatives

  More variables

  Linear approximation
Definition of the derivative

The derivative of a function \( f'(x) \) in one variable is

\[
  f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

(if the limit exists, in which case the function is differentiable).

Example

For \( f(x) = x^3 \), what is \( f'(x) \)?

There are at least two ways to think about the meaning:

- \( \frac{f(x + h) - f(x)}{h} \) is the slope of a secant line. In the limit, the secant line approaches the tangent line.

- The best linear approximation to \( f(a) \) for \( a \) near \( x \) is
  \[
  f(a) \approx f(x) + (a - x)f'(x)
  \]
Lecture 14: Partial derivatives

Review: Derivatives in 1D

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Definition of partial derivatives

Definition
If $f(x, y)$ is a function of two variables, its partial derivatives are

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$ 

Examples

Let $f(x, y) = 2x + 3y$. Then

$$f_x(1, 1) = \lim_{h \to 0} \frac{[2(1 + h) + 3(1)] - [2(1) + 3(1)]}{h} = 2$$
$$f_y(1, 1) = \lim_{h \to 0} \frac{[2(1) + 3(1 + h)] - [2(1) + 3(1)]}{h} = 3.$$ 

$f(x, y) = x^2 y$.
Definition of partial derivatives

Definition
If \( f(x, y) \) is a function of two variables, its partial derivatives are

\[
\begin{align*}
\frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \\
\frac{\partial f}{\partial y}(x, y) &= \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.
\end{align*}
\]

Exercise
Let \( f(x, y) = x + y^2 \). Compute (from the definition) \( f_x(1, 1) \) and \( f_y(1, 1) \).
What does it mean?
Remember one technique for plotting the surface \( z = f(x, y) \): slice.
Pick a value of \( y \) (say, 2), and look at the function \( g(x) = f(x, 2) \).
This is a function of one variable, and it's easier to plot ... or differentiate.

Example
For \( f(x, y) = x^2 - y^3 \), what does \( f_x(0.3, 0.5) \) mean?

Answer
- It's the slope of the tangent line to the \( x \)-slice: \( g(x) = f(x, 0.5) \) at \( x = 0.3 \).
- It's the slope of the parametric curve
  \( \vec{r}(t) = (t, 0.5, f(t, 0.5)) = (t, 0.5, t^2 - (0.5)^3) \) at \( t = 0.3 \).
Slicing

What does it mean?
Remember one technique for plotting the surface $z = f(x, y)$: slice. Pick a value of $y$ (say, 2), and look at the function $g(x) = f(x, 2)$. This is a function of one variable, and it's easier to plot . . . or differentiate.

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Slicing

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▶ It’s the slope of the parametric curve
  $\vec{r}(t) = (t, 0.5, f(t, 0.5)) = (t, 0.5, t^2 - (0.5)^3)$ at $t = 0.3$. 
Take another look at the definition of partial derivatives:

\[ f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \]

This says: compute the derivative of \( f \) as a function of \( x \), treating \( y \) as a constant.

**Examples**

\[
\begin{align*}
\text{\( f(x, y) = x^2 + y^3 \)} & \quad \text{\( f(x, y) = xy \)} & \quad \text{\( f(x, y) = xe^{y^2 \sin(y)^{10}} \)}
\end{align*}
\]

**Exercises**

Compute the partial derivatives:

\[
\begin{align*}
\text{\( f(x, y) = x^2 y^3 \)} & \quad \text{\( f(x, y) = e^x \cos(y) \)} & \quad \text{\( f(x, y) = x/y \) (for \( y \neq 0 \))}
\end{align*}
\]
Alternate notations

Derivatives of functions in one variable have lots of notations:

\[ \frac{df}{dx}, \quad \frac{d}{dx} f, \quad f', \quad f^{(1)}, \quad \dot{f} \]

So do partial derivatives:

\[ \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x} f, \quad f_x, \quad D_x f, \quad \partial_x f, \quad f_1, \quad D_1 f, \quad \partial_1 f \]

\(\partial\) is read “partial”. The “1” refers to the first coordinate.

We’ll mostly use \(f_x\).

\[ \frac{\partial f}{\partial x} \]

is also common, but it is a little misleading: you can’t think of partial derivatives as a ratio of infinitesimals in a sensible way.
Lecture 14: Partial derivatives

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More variables

Linear approximation
Differentiating again

If \( f(x, y) \) is a function of two variables, \( f_x(x, y) \) is also a function of two variables. Differentiate it again!

**Example**

If \( f(x, y) = x^2 y^3 \), then

\[
\begin{align*}
   f_x(x, y) &= 2xy^3 \\
   (f_x)_x(x, y) &= 2y^3 \\
   (f_x)_y(x, y) &= 6xy^2.
\end{align*}
\]

**Notation**

\[
(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = D_{xy} f = D_{12} f = \ldots
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Differentiating again

If $f(x, y)$ is a function of two variables, $f_x(x, y)$ is also a function of two variables. Differentiate it again!

Example

If $f(x, y) = x^2 y^3$, then

$$f_x(x, y) = 2xy^3$$

$$(f_x)_x(x, y) = 2y^3$$

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Examples

Exercise
Compute $f_{xx}$, $f_{xy}$, $f_{yx}$, and $f_{yy}$ for

$$ f(x, y) = x^2 y^3 $$

$$ f(x, y) = e^x \cos(y) $$

$$ f(x, y) = \frac{x}{y} \text{ (for } y \neq 0) $$

We can also compute higher derivatives:

Exercise
Compute $f_{xxx}$, $f_{xxy}$, and $f_{yyy}$ for $f(x, y) = x^2 y^3$.

How many third-order partial derivatives are there in all? Fourth-order?
Equality of mixed partials: Clairaut's Theorem

**Theorem (Clairaut, 1743)**

If \( f(x, y) \) is a function of two variables, and \( f_{xy} \) and \( f_{yx} \) are both continuous near \((a, b)\), then

\[
f_{xy}(a, b) = f_{yx}(a, b).
\]

So we don’t need to remember the order to do derivatives in. (Phew!) The condition is necessary; see the homework!

**Question**

How many different third-order partial derivatives are there now?
Lecture 14: Partial derivatives

Review: Derivatives in 1D

Partial derivatives

Higher derivatives

► More variables

Linear approximation
Increasing the number of variables

We’ve been focusing on the case of two variables, but you can treat three or more variables the same way:

\[
\begin{align*}
  f_x(x, y, z) &= \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h} \\
  f_y(x, y, z) &= \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h} \\
  f_z(x, y, z) &= \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}
\end{align*}
\]

**Question**

How many different second-order partial derivatives of \(f(x, y, z)\) are there?

**Answer**

Six.
Increasing the number of variables

We’ve been focusing on the case of two variables, but you can treat three or more variables the same way:

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\begin{align*}
  f_x(x, y, z) &= \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h} \\
  f_y(x, y, z) &= \lim_{h \to 0} \frac{f(x, y + h, z) - f(x, y, z)}{h} \\
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▶ Linear approximation
Linear approximations

Notice we have not defined when a function $f(x, y)$ is differentiable at $(a, b)$. That is because a function can have all partial derivatives, and still be very badly behaved.

Remember the other meaning of the derivative: $f'(a)$ gives the best linear approximation to $f(x)$ for $x$ near $a$:

$$f(x) \approx f(a) + (x - a)f'(a).$$

More precisely:

**Theorem**

*If $f(x)$ is differentiable at $x = a$, then if $g(x)$ is the error in the approximation,*

$$g(h) = f(a + h) - (f(a) + hf'(a)),$$

*then $g(h)$ goes to zero faster than linearly:*

$$\lim_{x \to a} \frac{g(h)}{h} = 0.$$
Approximations in two variables

What does this say about functions of two variables $f(x, y)$?

For fixed $b$, the slice $f(x, b)$ is a function of one variable. So

$$f(a + h, b) \approx f(a, b) + hf_x(a, b).$$

Or, slicing the other way,

$$f(a, b + k) \approx f(a, b) + kf_y(a, b).$$

What if we change both $x$ and $y$?

$$f(a + h, b + k) \approx$$

Add up the two contributions to the change of $f$.

Lots more next time!
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$$f(a + h, b + k) \approx f(a, b) + hf_x(a, b) + kf_y(a, b).$$

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