Lecture 13: Functions of two variables: Limits and continuity

October 21, 2010

Announcements

- Homework 6 due Tuesday, at beginning of class. 10% penalty if more than 5 minutes late.
- New TA: Alex Moll ⟨acm2141@columbia.edu⟩. Office hours: Monday 10:30–12:30, Math 406.
- Next office hours: Monday, 2:00–3:00, Math 614.
Lecture 13: Functions of two variables: Limits and continuity

- Contour maps

Limits and continuity
  - Pathologies
  - Definitions
  - Techniques
Contour maps

One common use of contour graphs: contour maps.

On the left is a contour (topographic) map near Somers, New York, about 45 miles NNE from Barnard. (At upper left is the Amawalk Reservoir, one of the reservoirs serving New York City.) Thick contours are drawn every 50 feet of elevation, thin contours are drawn every 10 feet.
Interpreting contour maps

- Closer together contour lines means a steeper slope.
- One trick is to figure out which way is up.
  - Look at the numbers, indicating elevation.
  - Closed loops almost always enclose hills, not dips. (A dip would usually be filled with water.)

If you’re working with a mathematical function, need to indicate which way is up somehow.

- What does a peak look like? A pass? A valley?
Lecture 13: Functions of two variables: Limits and continuity

▶ Contour maps

▶ Limits and continuity
  Pathologies
  Definitions
  Techniques
Pathologies

Consider the function $f(z) = \frac{x^2 - y^2}{x^2 + y^2}$.

This looks really weird.
Has a well-defined limit along every line through 0, but limits are different!
In polar coordinates:

$$f((r, \theta)_{\text{polar}}) = f(r \cos \theta, r \sin \theta)$$

$$= \frac{r^2(\cos(\theta))^2 - r^2(\sin(\theta))^2}{r^2(\cos(\theta))^2 + r^2(\sin(\theta))^2}$$

$$= \cos(\theta)^2 - \sin(\theta)^2$$

$$= \cos(2\theta).$$

Only depends on $\theta$!
Along each line through 0, there is a well-defined limit. No clear value to assign at 0.
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Limits: Intuition

Informal definition
A function $f(x, y)$ has a limit of $c$ at (say) $(2, 3)$ if we can make the value of $f$ approach $c$ by taking $x$ and $y$ close to (but not at) $(2, 3)$.

The function $f(x, y)$ is continuous (in some region) if it has a limit at every point (in the region), and the limit equals the value at the point.

Definitions are like 1-variable functions $f(x)$, but more things can go wrong.
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Limits: In theory

Definition (1)
For a function of two variables \( f(x, y) \), the limit of \( f(x, y) \) as \( (x, y) \) approach \( (x_0, y_0) \) is \( c \), written

\[
\lim_{(x,y) \to (x_0,y_0)} f(x, y) = c,
\]

if for every small interval \((c - \epsilon, c + \epsilon)\) around \( c \) there is a small disk \( D \) around \( (x_0, y_0) \) so that

\[
c - \epsilon < f(x, y) < c + \epsilon
\]

for every \((x, y)\) in \( D \), not equal to \((x_0, y_0)\).

Definition (2)
Alternatively, \( \lim_{(x,y) \to (x_0,y_0)} = c \) if for every sequence \( \{(x_n, y_n) \mid n = 1, 2, \ldots \} \) of points approaching \((x_0, y_0)\) (not equal to \((x_0, y_0)\)),

\[
\lim_{n \to \infty} f(x_n, y_n) = c.
\]
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For a function of two variables $f(x, y)$, the limit of $f(x, y)$ as $(x, y)$ approach $(x_0, y_0)$ is $c$, written

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Elementary functions

In practice, the formal definition above ("\(\varepsilon-\delta\)") is hard to work with. Let’s see some practical techniques!

Building blocks

The coordinate functions \(x\) and \(y\) are continuous functions of \((x, y)\). So are constants.

Elementary functions

If \(f_1(x, y)\) and \(f_2(x, y)\) are continuous (or have a limit) at \((x_0, y_0)\), so are the sum, difference, and product:

\[
\begin{align*}
    f_1(x, y) + f_2(x, y) & \quad \text{and} \quad f_1(x, y) - f_2(x, y) & \quad \text{and} \quad f_1(x, y)f_2(x, y).
\end{align*}
\]

So is the quotient if \(f_2(x, y) \neq 0\):

\[
\frac{f_1(x, y)}{f_2(x, y)}
\]

Examples

5\(x^2 + y^6\) or \(\frac{x^2 - y^2}{x^2 + y^2}\) (at most points).
More general functions

Continuous functions
If \( g(x) \) is a (1-var) continuous function and \( f(x, y) \) is a (2-var) continuous function, so is

\[ g(f(x, y)). \]

Examples
\[ \cos(x^2 + y^2) \text{ or } |x^2 + y^2| \text{ or } e^y \cos(x). \]
These will cover 99% of the functions you will see in practice, but it’s worth seeing what can go wrong.
Limits on lines

What about techniques for showing a function is not continuous? Say, what if we (naively) get $0/0$ in limit? Unfortunately, nothing as nice as L’Hôpital’s rule works.

Lines

Suppose $f(x, y)$ is a function, and $C_1$ and $C_2$ are two lines passing through $(0, 0)$.

- If $f(x, y)$ does not have a limit as $(x, y)$ approach $(0, 0)$ along $C_1$, then $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.
- If limits along $C_1$ and $C_2$ both exist and are not equal, then $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

This also works for other points than $(0, 0)$. Compare: $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

Examples

\[
\frac{x^2 + y}{x^2 + y^2} \text{ or } \frac{x^2 - y^2}{x^2 + y^2} \text{ or } \frac{xy}{x^2 + y^2}.
\]
Limits on curves

Even if limits along each line exist and are equal, this is not enough to conclude the overall limit exists!

Example

\[
\frac{xy^2}{x^2 + y^4}
\]

Can consider more general curves than lines. In example: Compare limit along \( y = 0 \) and \( y = x^2 \).

(This can get hard, and you will get hints in any such problem.)
Polar coordinates and Squeeze Theorem

Sometimes switching to polar coordinates can help. The points \((r = 0, \theta)_{\text{polar}}\) all represent the origin, for any value of \(\theta\).

**Example**

\[
\frac{x^2 - y^2}{x^2 + y^2} \text{ or } \frac{x^4 - y^4}{x^2 + y^2}
\]

One last, very general technique:

**Squeeze Theorem**

If \(g_1(x, y) < f(x, y) < g_2(x, y)\), and \(g_1\) and \(g_2\) have the same limit at 0:

\[
\lim_{(x, y) \to (0, 0)} g_1(x, y) = \lim_{(x, y) \to (0, 0)} g_2(x, y),
\]

then \(f(x, y)\) has the same limit.

**Example**

\[
0 < \frac{x^4 + y^4}{x^2 + y^2} < \frac{x^4 + 2x^2y^2 + y^4}{x^2 + y^2} = x^2 + y^2.
\]