Lecture 12: More differential equations. Functions of two variables

October 19, 2010

Announcements

- Homework 5 due today.
- Next office hours: Wednesday, 2–3PM, Math 614.
Lecture 12: More differential equations. Functions of two variables

- More differential equations
  - Higher-order equations
  - Non-homogeneous equations

A taste of multi-variable functions
In general, to solve \( af''(x) + bf'(x) + cf(x) = 0 \):

- Suppose \( f(x) = e^{\alpha x} \).
- Get characteristic equation: \( a\alpha^2 + b\alpha + c = 0 \).
- Solve for \( \alpha \): get \( \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). Call solutions \( \alpha_1 \) and \( \alpha_2 \).
- Basic solutions are (usually) \( e^{\alpha_1 x} \) and \( e^{\alpha_2 x} \); general solution is \( Ae^{\alpha_1 x} + Be^{\alpha_2 x} \).
- Find \( A \) and \( B \).

Behavior depends on \( b^2 - 4ac \):

- \( b^2 - 4ac > 0 \): Now \( \alpha_1, \alpha_2 \) are real, distinct. Two different growths.
- \( b^2 - 4ac < 0 \): Take a square root of a negative number; two complex roots. Alternate basic solutions using sin and cos.
Physical interpretation: Damping

In our earlier pendulum example (and also for projectiles), ignored friction/air resistance. (Our pendulum kept swinging forever...)

Typical air resistance: Force opposite to velocity and proportional to it: $\vec{F}_{\text{fric}} = -b\vec{v}$.

Consider a mass on a spring. The spring has a spring constant $k$, and $\vec{F}_{\text{spring}} = -kr(t)$. (Here $r(t)$ is the displacement from “rest”.) Now get

$$mr''(t) = -br'(t) - kr(t)$$

$$mr'' + br' + kr = 0.$$

Cases:

- $b = 0$: **undamped**: two periodic solutions, no decay.
- $b > 0$, $b^2 - 4km < 0$: **underdamped**: periodic solutions with an exponential decay.
- $b^2 - 4km > 0$: **over damped**: Two exponentially decaying solutions.
Repeated roots: examples

What about the case $b^2 - 4ac = 0$?

**Example**

From the bonus problem: $f''(x) + 2f'(x) + f(x) = 0$

Characteristic equation: $\alpha^2 + 2\alpha + 1 = 0$. Root: $\alpha = -1$, repeated.

Ordinary solution: $f(x) = e^{-x}$. Extra solution: $f(x) = xe^{-x}$.

Check: $f'(x) = e^{-x} - xe^{-x}$ and $f''(x) = -2e^{-x} + xe^{-x}$.

$f'' + 2f' + f = (-2e^{-x} + xe^{-x}) + 2(e^{-x} - xe^{-x}) + xe^{-x} = 0$.

**Example**

Consider $f''(x) = 0$.

This one looks a little degenerate! You know already general solution: $f(x) = Ax + B$.

It does fit the form we have.

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Physical meaning

These equations are general! Any time you have a physical system near an equilibrium point, there will be some restoring force

$$F_{\text{restore}} \approx -cf(t)$$

(where $f(t)$ is the “distance” from equilibrium) and a friction force

$$F_{\text{fric}} \approx -bf'(t)$$

for an equation like

$$af''(t) = -bf'(t) - cf(t).$$

If we adjust the friction $b$ without changing the other parameters, highest rate of decay comes at critical damping: $b^2 - 4ac = 0$. 
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Terminology

**Exponential growth** The solution is growing exponentially, like $e^{at} \cos(bt)$ for some $a > 0$.

Usually not a physics situation...

Does not mean solution is growing rapidly!

**Exponential decay** The solution goes to zero exponentially, like $e^{-at} \cos(bt)$ for some $a > 0$.

**Damping** Solutions that decay.

**Undamped** Solutions neither grow nor shrink in amplitude (on average), like $\cos(t)$ or $e^{it}$. Can only be ideal limit of physical system.

**Underdamped** Damped (decaying), but with $b$ small enough so $b^2 - 4ac < 0$, so solutions oscillate.

**Overdamped** Damped, with $b$ large so $b^2 - 4ac > 0$. Goes “thud”.

**Critically damped** Between underdamped and overdamped. Best for shock absorbers.
Simple harmonic oscillator A physical system that behaves like
\[ f''(t) = -cf(t). \]
Examples: Pendulum, mass on string, almost anything that oscillates forever.

Damped harmonic oscillator A physical system that behaves like
\[ f''(t) = -bf'(t) - cf(t). \]
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More differential equations
Higher-order equations
Non-homogeneous equations

A taste of multi-variable functions
Higher order equations

We might have an equation with higher derivatives:

\[ f'''(x) + f''(x) + f'(x) + f(x) = 1. \]

Solve it with similar techniques:

- Consider solutions of form \( f(x) = e^{\alpha x} \).
- Characteristic equation: \( \alpha^3 + \alpha^2 + \alpha + 1 = 0 \).
- Solutions: \( \alpha = -1, \alpha = i, \alpha = -i \).
- General solution: \( f(x) = Ae^{-x} + Be^{ix} + Ce^{-ix} \)
  or \( f(x) = Ae^{-x} + D \cos(x) + E \sin(x) \).

These equations turn out to be much less frequent.

(This material is not in the book, but it's easy.)
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- More differential equations
  - Higher-order equations
  - Non-homogeneous equations

- A taste of multi-variable functions
Non-homogeneous equations

Suppose we have an equation that looks like the second-order linear differential equations we’ve been studying, but with an extra term:

\[ f''(x) + 4f(x) = \cos(x). \]

Where do these arise?

- Driven pendulums
- Vertical spring

It is no longer true that if \( f_1(x) \) and \( f_2(x) \) are solutions, then \( f_1(x) + f_2(x) \) is a solution. If \( f_1 \) is a solution to equation above, and \( f_3 \) is a solution to the complementary equation

\[ f''(x) + 4f(x) = 0, \]

then \( f_1(x) + f_3(x) \) is a solution to original equation.
Solving non-homogeneous equations

To solve a non-homogeneous differential equation

\[ af''(x) + bf'(x) + cf(x) = G(x) \]

- Solve the characteristic equation \( af'' + bf' + cf = 0 \). Let the general solution be \( f_c(x) \).
- Find any one solution \( f_p(x) \) to original equation.
- General solution to original equation is \( f_p(x) + f_c(x) \).

Example

\( f''''(x) + 4f(x) = \cos(x) \)
Non-homogeneous equations: Simple cases

- \( G(x) \) a polynomial like \( x^2 \):
  
  Try \( f_p(x) \) a polynomial like \( Ax^2 + Bx + C \) of same or smaller degree.
  
  Example: \( f''(x) - 4f(x) = x^2 \).

- \( G(x) \) an exponential function like \( e^{\alpha x} \):
  
  Try \( f_p(x) \) an exponential \( Ae^{\alpha x} \) with same growth.
  
  Example: \( f''(x) + 3f'(x) + 2f(x) = e^x \).

- \( G(x) \) like \( \cos(kx) \):
  
  Try \( f_p(x) \) oscillating like \( A\sin(kx) + B\cos(kx) \), with the same period.
  
  Example: \( f'' + 4f(x) = \cos(x) \).
  
  (This is like previous case, with imaginary exponent.)

(The book goes into many more cases, but we won’t.)
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More differential equations
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▶ A taste of multi-variable functions
Some multi-variable functions

Functions of more than one variable are all around us.

Examples

- Altitude as a function of position.
- More?
One way of understanding functions of 2 variables is to draw the graph in 3D. Let’s see some examples. Try to figure out what the graph is before looking at the graphics.

- \( f(x, y) = x^2 + y^2 \).
- \( f(x, y) = \cos(x) \).
- \( f(x, y) = y^2 \cos(x)/10 \).
- \( f(x, y) = \cos(x + y) \).
- \( f(x, y) = \cos(x) + \cos(y) \).
- \( f(x, y) = \cos(x^2 + y^2) \).
Contour graphs

Another popular way to represent a 2D function \( f(x, y) \) (which requires less graphical skill) is to plot evenly-spaced traces:
Plot solutions to \( f(x, y) = a \) for (say) \( a = -1, -0.9, \ldots, 0.1, 0.0, 0.1, \ldots, 0.9, 1 \).
Lines are closer together when slope is steeper.
Contour maps

One common use of contour graphs: **contour maps**.
On the left is a contour (topographic) map near Somers, New York, about 45 miles NNE from Barnard. (At upper left is the Amawalk Reservoir, one of the reservoirs serving New York City.) Thick contours are drawn every 50 feet of elevation, thin contours are drawn every 10 feet.
Consider the function \( f(z) = \frac{x^2 - y^2}{x^2 + y^2} \).

This looks really weird.

Has a well-defined limit along every line through 0, but limits are different!

In polar coordinates:

\[
\begin{align*}
  f((r, \theta)_{\text{polar}}) &= f(r \cos \theta, r \sin \theta) \\
  &= \frac{r^2(\cos(\theta))^2 - r^2(\sin(\theta))^2}{r^2(\cos(\theta))^2 + r^2(\sin(\theta))^2} \\
  &= \frac{\cos(\theta)^2 - \sin(\theta)^2}{\cos(2\theta)} \\
  &= \cos(2\theta).
\end{align*}
\]

Only depends on \( \theta \)!

We’ll see more next time,