Complex division

We can also divide complex numbers:

\[
\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 - b^2 i^2} = \frac{a - bi}{a^2 + b^2}.
\]

What is this geometrically?
The number \(a - bi\) is called the complex conjugate of \(a + bi\), written \(\overline{a + bi}\).
Always have \(z \overline{z} = |z|^2\).

Complex numbers are unique: I

\(\mathbb{C}\) is \(\mathbb{R}\) plus a square root of \(-1\). Why not add some more elements?

Fundamental Theorem of Algebra

Any non-constant polynomial \(p(z)\) has a root in \(\mathbb{C}\), even if we allow complex coefficients.

This is very important, but a bit beyond us. The first (essentially) correct proof was due to Gauss in 1799, after more than 50 years of attempts. He gave at least four.
Why did the Greatest Mathematician give four proofs? Because it’s such a remarkable theorem. We added artificially a solution to \(x^2 + 1 = 0\), and suddenly we have solutions to everything!

Examples

- Find a solution to \(z^2 = i\).
- Find a solution to \(z^6 = -1\).
- Find all solutions to \(z^6 = 1\).
Complex numbers are unique: II

Can we find other nice products, besides those we already know about?

Definition

A vector product is a function that takes two vectors in $\mathbb{R}^n$ and gives another (in symbols: $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$) that is

- Distributive over addition: $\vec{u} \times (\vec{v} + \vec{w})$,
- Associative with scalar mult.: $(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$.

It has division if, for fixed $\vec{v}, \vec{w} \neq 0$, we can always solve $\vec{x} \times \vec{v} = \vec{w}$ for $\vec{x}$.

1, 2, 4, 8 Theorem [Frobenius,Hurwitz]

The only associative, commutative vector products with division are $\mathbb{R}$ and $\mathbb{C}$.

Drop commutative: Also have quaternions $\mathbb{H}$ (on $\mathbb{R}^4$).

Also drop associative: Still another: octonions $\mathbb{O}$ (on $\mathbb{R}^8$).

Quaternions are related to the cross product.

(Actually, need to assume “normed” in last case.)

Lecture 11: Complex numbers for differential equations

More complex arithmetic

- Complex numbers in calculus

Differential equations: Intro

Solving second-order linear differential equations

Unifying formulas

You know how to integrate $1/(x^2 - 1)$:

$$\int \frac{1}{x^2 - 1} \, dx = \int \left( \frac{1/2}{x - 1} - \frac{1/2}{x + 1} \right) \, dx = \frac{1}{2} \ln(x - 1) - \frac{1}{2} \ln(x + 1) + C.$$  

You also know how to integrate $1/(x^2 + 1)$:

$$\int \frac{1}{x^2 + 1} \, dx = \arctan(x) + C$$  

Complex numbers let you do these two problems in the same way:

$$\frac{1}{x^2 + 1} = \frac{i/2}{x + i} - \frac{i/2}{x - i}$$

$$\int \frac{1}{x^2 - 1} \, dx = \int \left( \frac{i/2}{x + i} - \frac{i/2}{x - i} \right) \, dx = \frac{1}{2} \ln(x + i) - \frac{1}{2} \ln(x - i) + C.$$  

These are actually the same, for correct generalization of $\ln$.  

Parametric curves in $\mathbb{C}$

Some parametric curves are convenient to express using complex numbers.

Example

The curve $z(t) = 2e^{it}$ is the circle around the origin of radius 2.

What’s the curve $z(t) = 2e^{it} + 1 + 3i$?

Example

The curve $z(t) = (1 + it)^2$ is $(1 - i^2, 2t)$, a parabola.
A differential equation is an equation for a function \( f(x) \), involving \( f \) and its derivatives.

**Examples**

- \( f'(x) = f(x) \). Solution: \( f(x) = e^x \).
- \( f''(x) = -f(x) \). Solutions: \( f(x) = \cos(x) \), \( f(x) = \sin(x) \).
- \( f'''(x) = x \). Solution:

Differential equations come up all the time!
Idea of differential equations

A differential equation is an equation for a function \( f(x) \), involving \( f \) and its derivatives.

Examples

- \( f'(x) = f(x) \). Solution: \( f(x) = e^x \).
- \( f''(x) = -f(x) \). Solutions: \( f(x) = \cos(x) \), \( f(x) = \sin(x) \).
- \( f'''(x) = x \). Solution: \( f(x) = \frac{x^4}{24} \).

Differential equations come up all the time!

Example: Pendulum

Question
Suppose you have a pendulum of length \( l \) and mass \( m \). What does its motion look like? What is angle \( \theta \) as a function of \( t \)?

Answer
Pendulum weight experiences a force \( gm \) downwards.
Tangential component: \( gm \sin(\theta) \), pointing inwards. Thus:

\[
ml\theta''(t) = -gm\sin(\theta(t))
\]

\[
\theta''(t) = -\frac{g}{l} \sin(\theta(t))
\]

\[
\approx -\frac{g}{l} \theta(t).
\]

(Why \( l\theta'' \)? That’s what the acceleration is. Also, needed for units.)
Solution (adopted from previous):

\[
\theta(t) = A\sin \left( \sqrt{\frac{g}{l}} t \right) + B \cos \left( \sqrt{\frac{g}{l}} t \right).
\]
Example: Pendulum, cont.

Pendulum’s motion is
\[ \theta(t) = A \cos(\sqrt{\frac{g}{l}}t) + B \sin(\sqrt{\frac{g}{l}}t). \]

Question
Suppose I pull the pendulum to an angle of .1 radian and let it go. Describe the motion.
Answer

Question
How long do the pendulum’s oscillations take?
Answer

Independent of size of swing! (As long as oscillations are small.)
Observed by Galileo; key to pendulum clocks.

Example: Pendulum, cont.

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Second-order linear differential equations

These types of equations come up often.
Definition
A second-order linear differential equation is an equation of the form
\[ a f''(x) + b f'(x) + c f(x) = 0. \]
(Second order: only second derivatives appear.)
Theorem
A second-order linear differential equation has two basic solutions. Any other solutions can be written as a linear combination of these two.
(Linear combination of \( f_1(x) \) and \( f_2(x) \): \( Af_1(x) + Bf_2(x) \), for constants \( A \) and \( B \).)
Idea: Position and first derivative determine the entire Taylor series.
But wait! What about . . .

We found four different solutions to $f''(t) = -f(t)$:

- $\cos(t)$, $\sin(t)$,
- $e^{it}$, $e^{-it}$.

What’s happening?

In fact, any two of these solutions can be taken as the basic solutions:

$$e^{it} = \cos(t) + i \sin(t)$$
$$e^{-it} = \cos(t) - i \sin(t)$$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$
$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

Basic solutions are not unique.

Lecture 11: Complex numbers for differential equations

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► Solving second-order linear differential equations

An example

Question

What are the solutions to $f''(t) + 2f'(t) + 5f(t) = 0$?

Answer

Guess that the answer might be $Ae^{\alpha t}$. Get

$$\alpha^2 A e^{\alpha t} + 2\alpha A e^{\alpha t} + 5A e^{\alpha t} = 0$$
$$\alpha^2 + 2\alpha + 5 = 0$$
$$\alpha = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

Two basic solutions:

$$f_1(t) = e^{(-1+2i)t} = e^{-t} e^{2it}$$
$$f_2(t) = e^{(-1-2i)t} = e^{-t} e^{-2it}$$

Alternatively:

$$g_1(t) = e^{-t} \cos(2t)$$
$$g_2(t) = e^{-t} \sin(2t)$$

Exercise: Solve $f''(t) - 2f'(t) + 2f(t) = 0$ with $f(0) = 1$, $f'(0) = 2$. 
An example

**Question**
What are the solutions to $f''(t) + 2f'(t) + 5f(t) = 0$?

**Answer**
Guess that the answer might be $Ae^{\alpha t}$. Get
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\[\alpha^2 + 2\alpha + 5 = 0\]
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Two basic solutions:
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**Exercise:** Solve $f''(t) - 2f'(t) + 2f(t) = 0$ with $f(0) = 1$, $f'(0) = 2$.

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Linear second-order equations: General case

In general, to solve $af''(x) + bf'(x) + cf(x) = 0$:

- Suppose $f(x) = e^{\alpha x}$.
- Get characteristic equation: $\alpha^2 + b\alpha + c = 0$.
- Solve for $\alpha$: get $\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$. Call solutions $\alpha_1$ and $\alpha_2$.
- Basic solutions are (usually) $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$; general solution is $Ae^{\alpha_1 x} + Be^{\alpha_2 x}$.
- Find $A$ and $B$ given whatever else you know.

Behavior depends on $b^2 - 4ac$:

- $b^2 - 4ac > 0$: Now $\alpha_1, \alpha_2$ are real, distinct. Two different growths.
- $b^2 - 4ac < 0$: Take a square root of a negative number; two complex roots. Alternate basic solutions using sin and cos.
- $b^2 - 4ac = 0$: Repeated root! Need another basic solution. Turns out to be $xe^{\alpha x}$, where $\alpha$ is root of characteristic equation.

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Physical interpretation: Damping

In our earlier pendulum example (and also for projectiles), ignored friction/air resistance. (Our pendulum kept swinging forever...)

Typical air resistance: Force opposite to velocity and proportional to it: $\vec{F}_{\text{fric}} = -b \vec{v}$.

Consider a mass on a spring. The spring has a spring constant $k$, and $F_{\text{spring}} = -kr(t)$. (Here $r(t)$ is the displacement from "rest"). Now get
\[mr''(t) = -br'(t) - kr(t)\]
\[mr'' + br' + kr = 0.\]

**Cases:**
- $b = 0$: undamped: two periodic solutions, no decay.
- $b > 0$, $b^2 - 4km < 0$: underdamped: periodic solutions with an exponential decay.
- $b^2 - 4km = 0$: critically damped.
- $b^2 - 4km > 0$: over damped: Two exponentially decaying solutions.

Critical damping turns out to decay the fastest. Useful in shock absorbers.