Lecture 10: Complex exponential

October 12, 2010

Announcements

- Homework 4 due today.
- Next office hours: Wednesday, 2–3PM, Math 614

Another product

Here’s another product, this one for 2D vectors.

\[(a, b) \ast (c, d) = (ac - bd, ad + bc).\]

It is not the dot product or the cross product. It looks a little weird. What can we say about it?

**Question**

What does multiplication by \((1, 0)\) do?

**Answer**

Nothing!

**Question**

What does multiplication by \((0, 1)\) do?

**Answer**

Rotate by 90° counterclockwise.

Exploring the product

Multiplication by \((1, 0)\) does nothing. Let's give \((1, 0)\) the temporary name "\(\langle 1 \rangle\)."

Multiplication by \((0, 1)\) rotates by 90° counterclockwise.
In particular, \((0, 1) \ast (0, 1) = (-1, 0) = -\langle 1 \rangle\): it's a square root of \(-1.
Let's give \((0, 1)\) the temporary name "\(\langle i \rangle\)."

Does this behave the way we expect?

Any 2D vector can be written \((x, y) = x\langle 1 \rangle + y\langle i \rangle\).

Then

\[(a, b) \ast (c, d) = (a\langle 1 \rangle + b\langle i \rangle) \ast (c\langle 1 \rangle + d\langle i \rangle)
= ac\langle 1 \rangle \ast \langle 1 \rangle + ad\langle 1 \rangle \ast \langle i \rangle + bc\langle i \rangle \ast \langle 1 \rangle + bd\langle i \rangle \ast \langle i \rangle
= ac\langle 1 \rangle + ad\langle i \rangle + bc\langle i \rangle - bd\langle 1 \rangle
= (ac - bd, ad + bd).\]

Everything was forced by properties: Commutativity, associativity with scalar multiplication, distributivity, plus what we knew about \((1, 0)\) and \((0, 1)\).

Properties of complex multiplication

**Theorem**

The product \(\ast\) defined above is

- Commutative: \(\vec{v} \ast \vec{w} = \vec{w} \ast \vec{v}\).
- Associative with scalars: \((a\vec{v}) \ast \vec{w} = a(\vec{v} \ast \vec{w}) = \vec{v} \ast (a\vec{w})\).
- Distributive: \(\vec{u} \ast (\vec{v} + \vec{w}) = \vec{u} \ast \vec{v} + \vec{u} \ast \vec{w}\).
- Associative: \(\vec{u} \ast (\vec{v} \ast \vec{w}) = (\vec{u} \ast \vec{v}) \ast \vec{w}\).
- Has a unit: \((1, 0) \ast \vec{v} = \vec{v}\).

Now let's drop the angle brackets on \(\langle 1 \rangle\) and \(\langle i \rangle\). Also, write the product just as multiplication.

We have seen a complex number \(x + yi\) can be thought of as a 2D vector, with product above. (Check: Is addition correct?)
Geometry of complex multiplication

Multiplication by $i$ (or $(0,1)$) had a nice geometric interpretation.

Question
Can we find a geometric interpretation for multiplication in general? What are some examples?

Answer
Yes! Similar triangles.

\[ 1 + i \star 1 + 0.5i = 0.5 + 1.5i \]

Exponentiation

There are three ways we will think about the exponential function:

- Compounded interest.
- Solution to a differential equation: \( \frac{d}{dt} e^t = e^t \).
- Taylor series: \( e^x = 1 + x + x^2/2 + x^3/6 + \cdots \).

What happens if we put in a complex argument?

All three methods will generalize.

At first we'll just treat this as a formal question, but it turns out to be very useful to think about the exponential of complex numbers.

Compounded interest

If I put $2000 in a savings account with an interest rate of 5\% for 1 year, I end up with...

- \( $2000(1 + 0.05) = $2100 \), compounded annually
- \( $2000(1 + 0.025)^2 = $2101.25 \), compounded twice a year
- \( $2000(1 + (0.05)/12)^{12} = $2102.32 \), compounded monthly
- \( $2000(1 + (0.05)/365)^{365} = $2102.53 \), compounded daily
- \( $2000 \exp(0.05) = $2102.54 \), compounded continuously

Theorem

\[ \lim_{n \to \infty} (1 + x/n)^n = e^x. \]

(e^x is also written \exp(x).)

Compounding imaginary interest

Now suppose our interest rate is \( i \). What happens then?

Compound the interest \( n \) times: compute \( (1 + i/n)^n \).

What if the interest rate is \( \pi i \)?

What do you guess the answer is? Do you want your money in that bank?)
Compounding imaginary interest

Now suppose our interest rate is $i$. What happens then?

Compound the interest $n$ times: compute 
\[(1 + i/n)^n].

What if the interest rate is $\pi i$?
What do you guess the answer is? Do you want your money in that bank?

$n = 4$

Compounding imaginary interest

Now suppose our interest rate is $i$. What happens then?

Compound the interest $n$ times: compute 
\[(1 + i/n)^n].

What if the interest rate is $\pi i$?
What do you guess the answer is? Do you want your money in that bank?

$n = 20$

Compounding imaginary interest

Now suppose our interest rate is $i$. What happens then?

Compound the interest $n$ times: compute 
\[(1 + i/n)^n].

What if the interest rate is $\pi i$?
What do you guess the answer is? Do you want your money in that bank?

$n = 1$

Compounding imaginary interest

Now suppose our interest rate is $i$. What happens then?

Compound the interest $n$ times: compute 
\[(1 + i/n)^n].

What if the interest rate is $\pi i$?
What do you guess the answer is? Do you want your money in that bank?

$n = 5$
Compounding imaginary interest

Now suppose our interest rate is $i$. What happens then?

Compound the interest $n$ times: compute $(1 + i/n)^n$.

What if the interest rate is $\pi i$?

What do you guess the answer is? Do you want your money in that bank?

Solving a differential equation

The geometric pictures suggested that $e^{\pi i} = -1$. Let's look at it from another point of view.

**Theorem**
The function $f(x) = e^x$ is the unique function that satisfies

- $f(0) = 1$ and
- $f'(x) = f(x)$.

More generally, for a constant $k$, the function $g(x) = e^{kx}$ satisfies $g'(x) = kg(x)$.

What about $h(t) = e^{it}$? It should satisfy $h'(t) = ih(t)$. What function satisfies this?

**Theorem**
The function $h(t) = \cos(t) + \sin(t) i$ satisfies

- $h'(t) = ih(t)$ and
- $h(0) = 1$.

We therefore define $e^{it}$ to be $\cos(t) + \sin(t) i$.

Taylor series

One final point of view is Taylor series: Expand out functions by a polynomial approximation.

**Theorem**

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!}$$

Check: This is compatible with $\frac{d}{dx} e^x = e^x$.

Other useful series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

Check: $\frac{d}{dx} \cos(x) = -\sin(x)$ and $\frac{d}{dx} \sin(x) = \cos(x)$.

Complex Taylor series

What happens if we compute $e^{ix}$?

$$e^{ix} = 1 + xi + \frac{(xi)^2}{2} + \frac{(xi)^3}{6} + \frac{(xi)^4}{24} + \frac{(xi)^5}{120} + \cdots$$

$$= 1 + xi - \frac{x^2}{2} - \frac{x^3 i}{6} + \frac{x^4}{24} + \frac{x^5 i}{120} + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^3 i}{6} + \frac{x^5}{120} + \cdots$$

$$+ \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots \right) i$$

$$= \cos(x) + \sin(x) i,$$

just like we got before.
Summary

**Theorem**

\[ e^{ix} = \cos(x) + \sin(x) i. \]

In particular,

\[ e^{\pi i} = -1. \]

This is called *Euler’s equation*. It has been called the most beautiful equation in mathematics!