Lecture 9: Velocity and acceleration

http://www.math.columbia.edu/~dpt/F10/CalcIII/

October 7, 2010

Announcements

- Homework 4 due Tuesday, October 12
- Next office hours: Monday, 10–11 AM (note change), Math 614
- Review the exponential function and Taylor series for Tuesday.

Velocity and speed

**Definition**

If \( \vec{r}(t) \) gives position of a moving object, the velocity of \( \vec{r}(t) \) at time \( t_0 \) is

\[
\vec{v}(t_0) = \lim_{s \to 0} \frac{\vec{r}(t_0 + s) - \vec{r}(t_0)}{s} = \frac{d\vec{r}}{dt}(t_0).
\]

The speed of \( \vec{r}(t) \) at time \( t_0 \) is

\[
\left| \frac{d\vec{r}}{dt}(t_0) \right|.
\]

**Example**

Suppose that \( \vec{r}(t) = (1, t, t^2) \). Then

\[
\vec{v}(t_0) = \lim_{h \to 0} \frac{(1, t_0 + h, (t_0 + h)^2) - (1, t_0, t_0^2)}{h} = (0, 1, 2t_0)
\]

\[
|\vec{v}(t_0)| = \sqrt{0 + 1 + 4t_0^2}.
\]

Differentiating vector functions

**Theorem**

Suppose \( \vec{r}(t) = (x(t), y(t), z(t)) \). Then

\[
\frac{d\vec{r}}{dt}(t_0) = (x'(t_0), y'(t_0), z'(t_0)).
\]

**Theorem**

If \( \vec{r}(t), \vec{s}(t) \) are differentiable vector functions,

\[
\frac{d(a\vec{r}(t))}{dt} = a\vec{r}'(t), \quad \frac{d(\vec{r}(t) + \vec{s}(t))}{dt} = \vec{r}'(t) + \vec{s}'(t)
\]

\[
\frac{d(a(t)\vec{r}(t))}{dt} = a'(t)\vec{r}(t) + a(t)\vec{r}'(t), \quad \frac{d(\vec{r}(t) \cdot \vec{s}(t))}{dt} = \vec{r}'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}'(t)
\]

...
More velocity and speed

Example
What is the speed at time $t = 1$ of
\[ \vec{r}(t) = (\sin(\pi t), 2\cos(\pi t), t^3)? \]

Answer

Mantra
Velocity is a vector. Speed is a number.

Mantra
The velocity vector points in the direction of the tangent line.

Circular motion

An important special case: An object moving in a circle.
\[ \vec{r}(t) = (\cos(t), \sin(t)) \]
\[ \vec{r}'(t) = (-\sin(t), \cos(t)) \]
\[ |\vec{r}'(t)| = 1 \]

In extended version, $R$ is the radius, $Rk$ is the speed.

Practical question
If I'm swinging a ball around my head and let go when string points to my left, where do you not want to be standing?

Newton's first law

Practical question
If I'm swinging a ball around my head and let go when string points to my left, where do you not want to be standing?

Lex 1
Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Law 1
Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

That is, unless something else happens, the velocity of a moving object is constant.

Acceleration

Definition
If $\vec{r}(t)$ gives position of a moving object, the velocity of $\vec{r}(t)$ at time $t$ is
\[ \vec{v}(t) = \vec{r}'(t). \]

The acceleration is
\[ \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t). \]

Example
What is the acceleration of
\[ \vec{r}(t) = (\sin(\pi t), 2\cos(\pi t), t^3)? \]

Answer
\[ \vec{r}'(t) = (\pi \cos(\pi t), -2\pi \sin(\pi t), 3t^2) \]
\[ \vec{r}''(t) = (-\pi^2 \sin(\pi t), -2\pi^2 \cos(\pi t), 6t). \]
Circular motion, reprise

What’s the acceleration of an object moving in a circle?

\[ \vec{r}(t) = (R \cos(kt), R \sin(kt)) \]
\[ \vec{v}(t) = \vec{r}'(t) = (-Rk \sin(kt), Rk \cos(kt)) \]
\[ |\vec{v}(t)| = Rk \]
\[ \vec{a}(t) = \vec{r}''(t) = (-Rk^2 \cos(kt), -Rk^2 \sin(kt)) \]
\[ |\vec{r}''(t)| = Rk^2 = |\vec{v}(t)|^2 / R. \]

The acceleration always points to the center of the circle.

Newton’s second law

Lex 2
Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Law 2
The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.

That is, if \( \vec{a} \) is the acceleration and \( \vec{F} \) is the force,

\[ \vec{F} = m \vec{a} \]

for some constant \( m \) (the mass).

For instance, when swinging a ball in a circle, the force is happening along the string (pointing to the center), with magnitude

\[ \frac{m|\vec{v}|^2}{R}. \]

Normal and tangential acceleration

We can break up the acceleration into components parallel and perpendicular to the velocity.

Exercise
If the speed of a particle is constant, the acceleration is perpendicular to velocity. (HW 13.4.22)

Component of acceleration perpendicular to velocity changes direction, leaving speed unchanged.

Component of acceleration parallel to velocity changes speed, leaving direction unchanged.

Question
How do I speed up the ball on a string?

Motion in gravity

On the Earth’s surface, gravity pulls downwards at a force of \( gm \), where \( g = 9.8 \, \text{m/s}^2 \).

We can find the velocity and position at later times by integration.

Example
Suppose I throw a ball upwards and eastwards from the ground with initial velocity \((5 \, \text{m/s}, 0, 0)\). When and where does it hit the ground again?

Answer
Set up coordinates so the origin is the initial point of the ball. Let \( \vec{r}(t) \) be its path. We know:

\[ \vec{r}(0) = (0, 0, 0) \quad \vec{r}'(0) = (5, 0, 5) \quad \vec{r}''(t) \approx (0, 0, -10). \]

Now integrate:

\[ \vec{r}(t) = (5t, 0, 5t - 5t^2) \quad \vec{r}'(t) = (5, 0, 5 - 10t) \]

The \( z \) coordinate of \( \vec{r}(t) \) is 0 again when \( t = 1 \), so the ball lands 1 second later, 5 meters east.

(We are ignoring air resistance.)
(Question 13.4.29) A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You can position catapults 100 m from the walls. You'll set fire to the town by catapulting heated rocks over the walls with initial velocity 80 m/s. At what angles should you tell your soldiers to set the catapult?

Question

If you were designing the catapult, how would you measure that speed of 80 m/s?