Lecture 5: Lines and planes in space, II

http://www.math.columbia.edu/~dpt/F10/CalcIII/

September 21, 2010
Announcements

- Homework 2 is due.
- Thanks for the feedback!
  - Will keep printed handouts.
  - Font size is slightly larger.
  - Color
  - Will give more chance to think.
  - Writing on board: Conflicting opinions. Slides cannot contain all the material.
  - More feedback always welcome.

- Sorry: sick.
Lecture 5: Lines and planes in space, II

► Reprise

Degrees of freedom

Planes in space (implicit)

Implicit lines and parametric planes in space

Problem solving techniques
Lines in the plane

Implicit equations
Represent a line by giving an equation:
Dot product with normal vector is fixed.

\[
\{(x, y) \mid 2x + 3y = 6\} = \\
\{(x, y) \mid (2, 3) \cdot (x, y) = 6\}
\]

General form:
\[
\{(x, y) \mid \vec{n} \cdot (x, y) = \vec{n} \cdot \vec{r}_0\}
\]

Parametric representation
Represent a line by listing the points as a parameter varies.

\[
\{(3t, 2 - 2t) \mid t \in \mathbb{R}\} = \\
\{(0, 2) + t(3, -2) \mid t \in \mathbb{R}\}
\]

General form:
\[
\{\vec{r}_0 + t\vec{v} \mid t \in \mathbb{R}\}
\]
We generalize the parametric representation of lines.

\[ \{ \vec{r}_0 + t\vec{v} \mid t \in \mathbb{R} \} \]

\( \vec{r}_0, \vec{v} \) parameters

Two lines are parallel if direction vector for one is a multiple of direction vector for other.

Two lines that are not parallel may intersect.

If two lines do not intersect and are not parallel, they are skew.

If they do intersect, we can find the angle between them by using the dot product:

Line 1: \( \{ \vec{r}_1 + t\vec{v}_1 \mid t \in \mathbb{R} \} \)

Line 2: \( \{ \vec{r}_2 + t\vec{v}_2 \mid t \in \mathbb{R} \} \)

\[
\cos(\theta) = \frac{|\vec{v}_1 \cdot \vec{v}_2|}{\|\vec{v}_1\| \|\vec{v}_2\|}
\]

(Note: Angle should be between 0 and \( \pi/2 \).)
Lines in space: Example

Question
Do the two lines below intersect? What’s the angle?

\[
\{(1,2,3) + t(1,1,1) \mid t \in \mathbb{R}\}
\]
\[
\{(2,2,2) + t(-2,0,2) \mid t \in \mathbb{R}\}
\]

Answer
To find whether they intersect, write out the equations, using different parameters for the two lines:

\[
x = 1 + t \\
y = 2 + t \\
z = 3 + t
\]
\[
x = 2 - 2s \\
y = 2 \\
z = 2 + 2s
\]

There is a solution with \(t = 0, s = 0.5\). So the lines do intersect.

Angle is determined by dot product:

\[
\cos(\theta) = \frac{(1,1,1) \cdot (-2,0,2)}{\| (1,1,1) \| \| (-2,0,2) \|} = 0.
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Lines in space: Example

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Do the two lines below intersect? What’s the angle?

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\end{align*}

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Lecture 5: Lines and planes in space, II

Reprise

 Degrees of freedom

Planes in space (implicit)

Implicit lines and parametric planes in space

Problem solving techniques
Counting degrees of freedom

Will represent planes in space like implicit rep. of lines. E.g.,

\[ \{(x, y, z) \mid x + y + z = 1\} . \]

Why does this represent a plane?
The number of **degrees of freedom** of a system of equations is the dimension of the space of solutions.

**General rule**
Number of degrees of freedom = (number of variables) − (number of equations).

**Example**

\[ \{(x, y) \mid x + y = 1\} \quad \text{Degrees of freedom} = 2 − 1 = 1 \quad \text{Describes a line} \]

\[ \{(x, y, z) \mid x + y + z = 1\} \quad \text{Degrees of freedom} = 3 − 1 = 2 \quad \text{Describes a plane} \]
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Degrees of freedom: Meaning

General rule
Number of degrees of freedom = (number of variables) − (number of equations).

If number of degrees of freedom (d.o.f.) is
- 1, expect a line (or other curve)
- 2, expect a plane (or other surface)
- 0, expect a point (or collection of points)
- -1, expect no solutions.

But there are exceptions. Equations may be
- Redundant: One equation is determined by others.
- Inconsistent: There are no solutions.
Degrees of freedom: Meaning

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- Redundant: One equation is determined by others.
- Inconsistent: There are no solutions.
Degrees of freedom: Example

**General rule**
Number of degrees of freedom = (number of variables) − (number of equations).

**Example**
Intersection of two lines.

\[
\begin{align*}
  x &= 1 + t \\
  y &= 2 + t \\
  z &= 3 + t \\
  x &= 2 - 2s \\
  y &= 2 \\
  z &= 2 + 2s
\end{align*}
\]

Number of variables is 5 (\(x, y, z, t\)).
Number of equations is 6.
Expected d.o.f. is \(-1\). But there is a solution!
Equations are redundant, since lines do intersect (not skew).
Vector equations

A vector equation counts for as many equations as the dimension.

Example

Same intersection:

\[(1, 2, 3) + t(1, 1, 1) \mid t \in \mathbb{R}\} \cap \{(2, 2, 2) + s(-2, 0, 2) \mid s \in \mathbb{R}\}

\[(1, 2, 3) + t(1, 1, 1) = (2, 2, 2) + s(-2, 0, 2)\]

This counts as three equations: Equate each of the components.

Expected degrees of freedom = \(2 - 3 = -1\) (but actually is 0).

General intersection of lines:

\[\{\vec{r}_1 + t\vec{v}_1 \mid t \in \mathbb{R}\} \cap \{\vec{r}_2 + s\vec{v}_2 \mid s \in \mathbb{R}\}\]

\[\vec{r}_1 + t\vec{v}_1 = \vec{r}_2 + s\vec{v}_2\]

Expected degrees of freedom = \(2 - 3 = -1\) (but is 0 if lines intersect).

Note: \(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2\) are fixed, not variables.
Lecture 5: Lines and planes in space, II

Reprise

Degrees of freedom

- **Planes in space (implicit)**

  Implicit lines and parametric planes in space

  Problem solving techniques
Planes in space

For planes in space, most convenient is a generalization of implicit rep. of lines.

Principle

Set of vectors perpendicular to fixed vector is a plane (through $\vec{0}$).

Fix vector $\vec{n} = (a, b, c)$. Plane is

$$\{ \vec{r} \mid \vec{n} \cdot \vec{r} = 0 \} = \{(x, y, z) \mid ax + by + cz = 0 \}$$  (Passing through $\vec{0}$)

$$\{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \} = \{(x, y, z) \mid ax + by + cz = s \}$$  (General)

Main problem is often to find the normal vector.

Question

What is the plane through $P = (1, 1, 0)$, $Q = (0, 1, 1)$, and $R = (1, 0, 1)$?

Answer

The vectors $\vec{PQ} = (-1, 0, 1)$ and $\vec{PR} = (0, -1, 1)$ lie on the plane.

Normal vector is perpendicular to both: $\vec{n} = \vec{PQ} \times \vec{PR} = (1, 1, 1)$.

Plane is $\{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \} = \{ \vec{r} \mid \vec{n} \cdot \vec{r} = 2 \} = \{(x, y, z) \mid x + y + z = 2 \}$. 
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Planets in space

For planes in space, most convenient is a generalization of implicit rep. of lines.

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Main problem is often to find the normal vector.

**Question**

What is the plane through $(6, 0, -2)$ and containing the line

$(x, y, z) = (4 - 2t, 3 + 5t, 7 + 4t)$?
Planes in space

For planes in space, most convenient is a generalization of implicit rep. of lines.

**Principle**

Set of vectors perpendicular to fixed vector is a plane (through $\vec{0}$).

Fix vector $\vec{n} = (a, b, c)$. Plane is

$$\{\vec{r} | \vec{n} \cdot \vec{r} = 0\} = \{(x, y, z) | ax + by + cz = 0\} \quad \text{(Passing through } \vec{0})$$

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Main problem is often to find the normal vector.

**Question**

What is the plane through $(6, 0, -2)$ and containing the line $(x, y, z) = (4 - 2t, 3 + 5t, 7 + 4t)$?

**Answer**

Two vectors parallel to the plane: $(-2, 5, 4)$ and $(4, 3, 7) - (6, 0, -2) = (-2, 3, 9)$. Normal: cross product $(33, 10, 4)$. Equation: $\vec{r} \cdot (33, 10, 4) = (6, 0, -2) \cdot (33, 10, 4) = 190$. 
Planes in space: Angles

Angle between lines in $\mathbb{R}^2$ is angle between dir. vectors or between normal vectors.

\[
\{ \vec{r}_1 + t\vec{v}_1 \mid t \in \mathbb{R} \} \quad \text{Direction vector } \vec{v}_1 = (a_1, b_1) \\
\{ \vec{r}_2 + t\vec{v}_2 \mid t \in \mathbb{R} \} \quad \text{Direction vector } \vec{v}_2 = (a_2, b_2) \\
\text{Normal vector } \vec{n}_1 = (-b_1, a_1) \quad \text{Normal vector } \vec{n}_2 = (-b_2, a_2)
\]

\[
\cos(\theta) = \frac{\vec{v}_1 \cdot \vec{v}_2}{\| \vec{v}_1 \| \| \vec{v}_2 \|} = \frac{\vec{n}_1 \cdot \vec{n}_2}{\| \vec{n}_1 \| \| \vec{n}_2 \|}
\]

For angles between planes in $\mathbb{R}^3$, use angle between normal vectors.

\[
\{ \vec{r} \in \mathbb{R}^3 \mid \vec{n}_1 \cdot \vec{r} = \vec{n} \cdot \vec{r}_1 \} \quad \text{Normal vector } \vec{n}_1 \\
\{ \vec{r} \in \mathbb{R}^3 \mid \vec{n}_2 \cdot \vec{r} = \vec{n} \cdot \vec{r}_2 \} \quad \text{Normal vector } \vec{n}_2
\]

\[
\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{\| \vec{n}_1 \| \| \vec{n}_2 \|}
\]

- Parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$ (i.e., $\vec{n}_2 = a\vec{n}_1$)
- Perpendicular if $\vec{n}_1$ is perpendicular to $\vec{n}_2$ (i.e., $\vec{n}_1 \cdot \vec{n}_2 = 0$)
Planes in space: Angles, cont.

For angle between line and plane in $\mathbb{R}^3$, look at angle between

- direction vector $\vec{v}_L$ of line and
- normal vector $\vec{n}_P$ of plane.

- **Perpendicular** if $\vec{v}_L$ is parallel to $\vec{n}_P$.
- **Parallel** if $\vec{v}_L$ is perpendicular to $\vec{n}_P$.
- In general,

$$\cos(\theta + \pi/2) = \frac{\vec{v}_L \cdot \vec{n}_P}{\|\vec{v}_L\| \|\vec{n}_P\|}.$$
Planes in space: Angles, cont.

For angle between line and plane in $\mathbb{R}^3$, look at angle between

- direction vector $\vec{v}_L$ of line and
- normal vector $\vec{n}_P$ of plane.

- *Perpendicular* if $\vec{v}_L$ is parallel to $\vec{n}_P$.
- *Parallel* if $\vec{v}_L$ is perpendicular to $\vec{n}_P$.

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For angle between line and plane in $\mathbb{R}^3$, look at angle between
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Planes in space: Angles, cont.

For angle between line and plane in $\mathbb{R}^3$, look at angle between

- direction vector $\vec{v}_L$ of line and
- normal vector $\vec{n}_P$ of plane.

- **Perpendicular** if $\vec{v}_L$ is parallel to $\vec{n}_P$.
- **Parallel** if $\vec{v}_L$ is perpendicular to $\vec{n}_P$.
- In general,

\[
\cos(\theta + \pi/2) = \frac{\vec{v}_L \cdot \vec{n}_P}{\|\vec{v}_L\| \|\vec{n}_P\|}.
\]
Distance to lines and planes

For a line in $\mathbb{R}^2$, distance is given by dot product with normal:

$$L = \{ \vec{r} \in \mathbb{R}^2 \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}$$

$$\text{dist}(\vec{p}, L) = \text{comp}_{\vec{n}}(\vec{p} - \vec{r}_0)$$

$$= \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}$$

For a plane in $\mathbb{R}^3$, distance is given by dot product with normal:

$$P = \{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}$$

$$\text{dist}(\vec{p}, P) = \text{comp}_{\vec{n}}(\vec{p} - \vec{r}_0)$$

$$= \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}$$

In both cases, sometimes easier to look at unit normal vector $\frac{\vec{n}}{\|\vec{n}\|}$.

Question

What’s the distance from $(5, 6, 7)$ to the plane through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$?

Answer

We computed the normal vector earlier: $\vec{n} = (1, 1, 1)$.

Distance $= \text{comp}_{\vec{n}}((5, 6, 7) - (1, 0, 0)) = \frac{(1, 1, 1) \cdot (4, 6, 7)}{\|(1, 1, 1)\|} = \frac{17}{\sqrt{3}}$. 
Distance to lines and planes

For a line in \( \mathbb{R}^2 \), distance is given by dot product with normal:

\[
L = \{ \vec{r} \in \mathbb{R}^2 \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}
\]

\[
dist(\vec{p}, L) = \text{comp}_\vec{n}(\vec{p} - \vec{r}_0) = \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}
\]

For a plane in \( \mathbb{R}^3 \), distance is given by dot product with normal:

\[
P = \{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}
\]

\[
dist(\vec{p}, P) = \text{comp}_\vec{n}(\vec{p} - \vec{r}_0) = \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}
\]

In both cases, sometimes easier to look at unit normal vector \( \frac{\vec{n}}{\|\vec{n}\|} \).

Question

What’s the distance from \((5, 6, 7)\) to the plane through \((1, 0, 0)\), \((0, 1, 0)\), and \((0, 0, 1)\)?

Answer

We computed the normal vector earlier: \( \vec{n} = (1, 1, 1) \).

Distance = \( \text{comp}_\vec{n}((5, 6, 7) - (1, 0, 0)) = \frac{(1, 1, 1) \cdot (4, 6, 7)}{\|(1, 1, 1)\|} = \frac{17}{\sqrt{3}} \).
Lecture 5: Lines and planes in space, II

Reprise

Degrees of freedom

Planes in space (implicit)

Implicit lines and parametric planes in space

Problem solving techniques
Implicit lines

We can also give equations for lines in space, if we like. Now we need two equations.

$$\{(2, 2, 2) + s(-2, 0, 2) \mid t \in \mathbb{R}\}$$

$$x = 2 - 2t\quad y = 2\quad z = 2 + 2t$$

Eliminate $t$ from the equations by $t = \frac{2-x}{2}$:

$$y = 2\quad z = 2 + 2\left(\frac{2-x}{2}\right) = 4 - x.$$  

More generally:

$$x = x_0 + at\quad y = y_0 + bt\quad z = z_0 + ct$$

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Book calls this the symmetric form of line. But please don't memorize it.
We can also treat planes parametrically. Now we need two parameters:

\[
\{(x, y, z) \mid x + y + z = 1\} = \{(1, 0, 0) + t(-1, 1, 0) + s(-1, 0, 1) \mid s, t \in \mathbb{R}\}
\]

\[
= \{\vec{r}_0 + t\vec{v}_1 + s\vec{v}_2 \mid s, t \in \mathbb{R}\}
\]

Idea: start at point \((1, 0, 0)\), step \(t\) units in one direction and \(s\) units in another. We won’t do much with this now, but this is useful if (e.g.) you want to describe a square or a triangle in space: Can restrict range of \(s\) and \(t\).
Lecture 5: Lines and planes in space, II

Reprise

Degrees of freedom

Planes in space (implicit)

Implicit lines and parametric planes in space

► Problem solving techniques
Problem solving techniques

There are lots of things you can do with normal vectors and direction vectors, probably too many for you to memorize.

Principles

- If looking for a line: what is the direction vector of the line? Maybe you can find it directly, or maybe you can say something about it.
- If looking for a plane: What is the normal vector? Again, maybe you can only say something about it; maybe you know it’s perpendicular to something else.
- If looking for a distance: Is there a vector whose length is the distance?
- If looking for an angle: What two vectors have an angle equal to what you’re looking for?
Some challenge problems

**Question**
What is the line of intersection between the planes \(x + y + z = 1\) and \(x - 2y + 3z = 3\) in parametric form?
(Find the direction vector. What is it perpendicular to?)

**Question**
What is the distance between the lines \((x, y, z) = (1 + t, 2 + t, 3 + t)\) and \((x, y, z) = (3 + t, 3, 3 - t)\)?
(Find a vector whose length is the distance.)
Announcements

- Homework 2 is due.
- Thanks for the feedback!
  - Will keep printed handouts.
  - Font size is slightly larger.
  - Color
  - Will give more chance to think.
  - Writing on board: Conflicting opinions. Slides cannot contain all the material.
  - More feedback always welcome.
- Sorry: sick.