Lecture 4: Lines and planes in space

http://www.math.columbia.edu/~dpt/F10/CalcIII/

September 16, 2010
Announcements

- HW1 has been graded. Pick up in drop box, location TBA.
- HW1 solutions on CourseWorks (under “Class Files”).
- Homework due on Tuesday.
- No late homeworks. 10% penalty for homeworks more than 5 minutes late in class. (You may also put them in drop box on 4th floor of Math building by 12:30.)
- No office hours on Monday.
- Printed copies of slides are distributing.
Lecture 4: Lines and planes in space

▶ Cross products

Lines in the plane

Lines in space (parametric)

Planes in space (implicit)
Cross product recap

The cross product takes two vectors in $\mathbb{R}^3$ and gives another vector in $\mathbb{R}^3$.

\[
(v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)
\]

\[
\det(\vec{u}, \vec{v}, \vec{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1)
\]

\[
\vec{u} \cdot (\vec{v} \times \vec{w}) = \det(\vec{u}, \vec{v}, \vec{w}) = \text{Area of parallelepiped with sides } \vec{u}, \vec{v}, \vec{w}
\]

**Theorem**

If $\vec{v}$ and $\vec{w}$ form an angle $\theta$ (with $\theta > 0$), then $\vec{v} \times \vec{w}$ is

1. perpendicular to $\vec{v}$ and $\vec{w}$,
2. has length $\|\vec{v}\|\|\vec{w}\| \sin(\theta)$,
3. in direction given by right hand rule.

**Right hand rule:** If (right hand) fingers curl from $\vec{v}$ to $\vec{w}$, thumb points at $\vec{v} \times \vec{w}$.

Compare: $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$. 
Cross product: Geometric proof

Theorem
If \( \vec{v} \) and \( \vec{w} \) form an angle \( \theta \), then \( \vec{v} \times \vec{w} \) is
1. perpendicular to \( \vec{v} \) and \( \vec{w} \),
2. has length \( \| \vec{v} \| \| \vec{w} \| \sin(\theta) \),
3. in direction given by right hand rule.

Proof (except for right hand rule).

\[
\det(\vec{u}, \vec{v}, \vec{w}) = \vec{u} \cdot (\vec{v} \times \vec{w}) = \pm (\text{Vol. of parallelepiped with sides } \vec{u}, \vec{v}, \vec{w})
\]

\[
(\text{Vol. of parallelepiped}) = (\text{Area of base})(\text{Height})
\]

\[
(\text{Area of base}) = \| \vec{v} \| \| \vec{w} \| \sin(\theta)
\]

\[
(\text{Height}) = |\text{comp}_{\vec{n}}(\vec{u})| = |\vec{u} \cdot \vec{n}|
\]

where \( \vec{n} \) is unit vector perpendicular to base (unit normal vector). Must then have

\[
\vec{v} \times \vec{w} = |\text{Area of base}| \vec{n}.
\]
Properties of cross product

A product of two basis vectors is $\pm$ the third basis vector.

\[
\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{i} = -\vec{k},
\]

\[
\vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{j} = -\vec{i},
\]

\[
\vec{k} \times \vec{i} = \vec{j}, \quad \vec{i} \times \vec{k} = -\vec{j}.
\]

In fact, this is enough to reconstruct formula for cross product:

\[
\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).
\]

Cross product does not satisfy all the properties you expect:

\[
\vec{v} \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u} \quad \text{(Distributive)}
\]

\[
(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})
\]

\[
\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \neq \vec{w} \times \vec{v} \quad \text{(Anti-commutative)}
\]

\[
\vec{v} \times (\vec{w} \times \vec{u}) \neq (\vec{v} \times \vec{w}) \times \vec{u} \quad \text{(Not associative)}
\]

Anti-commutative means that the sign changes when you switch the two factors. You probably haven’t seen this before! In particular,

\[
\vec{v} \times \vec{v} = 0.
\]
Lecture 4: Lines and planes in space

Cross products

Lines in the plane

Lines in space (parametric)

Planes in space (implicit)
We will now take a look at how to represent lines and planes.

**Question**
Why is this in a calculus class?

**Answer**
Derivatives!
Approximate \( f(x) \) near \( x_0 \) by
\[
f(x) \approx f(x_0) + f'(x_0)(x - x_0)
\]
Lines in the plane: Implicit equations

Typical representation of lines in plane:
\[ y = ax + b \quad a, b \text{ parameters} \]

Doesn’t work for vertical lines. More general:
\[ L = \{(x, y) \mid px + qy = s\} \quad p, q, s \text{ parameters} \]
\[ = \{\vec{r} \mid \vec{n} \cdot \vec{r} = s\} \quad \vec{r} = (x, y), \vec{n} = (p, q) \}

Vector \( \vec{n} \) is called a normal vector to \( L \); it is perpendicular to \( L \).
Called an implicit representation, since we don’t explicitly give the points on the line.

\[ y = -(2/3)x + 2 \]
Typical representation of lines in plane:

\[ L = \{(x, y) \mid y = ax + b\} \quad a, b \text{ parameters} \]

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Called an implicit representation, since we don’t explicitly give the points on the line.

\[ \{(x, y) \mid 2x + 3y = 6\} \]
Typical representation of lines in plane:

\[ L = \{ (x, y) \mid y = ax + b \} \quad a, b \text{ parameters} \]

Doesn’t work for vertical lines. More general:

\[ L = \{ (x, y) \mid px + qy = s \} \quad p, q, s \text{ parameters} \]

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Vector \( \vec{n} \) is called a \textit{normal vector} to \( L \); it is perpendicular to \( L \).

Called an \textit{implicit} representation, since we don’t explicitly give the points on the line.

\[ \{ (x, y) \mid (x, y) \cdot (2, 3) = 6 \} \]
Can also represent a line by listing points.

\[ L = \{(x_0 + ta, y_0 + tb \mid t \in \mathbb{R}\} \quad x_0, y_0, a, b \text{ parameters} \]
\[ = \{\vec{r}_0 + t\vec{v} \mid t \in \mathbb{R}\} \quad \vec{r}_0 = (x_0, y_0), \quad \vec{v} = (a, b) \]

This is a \textit{parametric} representation; \( t \) is called the \textit{parameter}.
\( \vec{r}_0 + t\vec{v} \) means “start at \( \vec{r}_0 \), and go \( t \) units in direction \( \vec{v} \)’’.

**Question**

What’s a parametric representation for \( y \)-axis? What’s another?

Vector \( \vec{v} \) a \textit{direction vector} for \( L \); it is parallel to \( L \).

\[ \{(3t, 2 - 2t) \mid t \in \mathbb{R}\} \]
Lines in the plane: Parametric representation

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**Question**

What’s a parametric representation for \( y \)-axis? What’s another?

Vector \( \vec{v} \) a direction vector for \( L \); it is parallel to \( L \).

\[ \{(0, 2) + (3, -2)t \mid t \in \mathbb{R}\} \]
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### Lines in the plane: Parametric representation
**Question**
What is the line through \( P = (-1, 1) \) and \( Q = (2, 0) \)?

**Answer**
Direction vector is \( \vec{v} = \overrightarrow{PQ} = (2, 0) - (-1, 1) = (3, -1) \).
Choose basepoint \( \vec{r}_0 = (2, 0) \).
Parametric representation is
\[
\{ \vec{r}_0 + t\vec{v} \mid t \in \mathbb{R} \} = \{ (2 + 3t, 0 - t) \mid t \in \mathbb{R} \}
\]
Normal vector is \( \vec{n} = (1, 3) \).
Implicit representation is
\[
\{ \vec{r} \mid \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \} = \{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}
\]
\[
= \{ (x, y) \mid x + 3y = 2 \}.
\]
Lecture 4: Lines and planes in space

Cross products

Lines in the plane

▶ Lines in space (parametric)

Planes in space (implicit)
The representation of lines that generalizes best to lines in space (and higher) is the parametric one.

\[ L = \{ \vec{r}_0 + t\vec{v} | t \in \mathbb{R} \} \]

\( \vec{r}_0, \vec{v} \) parameters

**Example**

What’s a parametric equation for

- the line through \((1, 2, 3)\) in direction \((1, 1, 1)\)?
- the line segment between \((2, 2, 2)\) and \((0, 2, 4)\)?

(*Hint:* Find the line first. Which values of \(t\) are on the segment?)
Two lines are *parallel* if direction vector for one is a multiple of direction vector for other. Two lines that are not parallel may *intersect*. If two lines do not intersect and are not parallel, they are *skew*. If they do intersect, we can find the angle between them by using the dot product:

\[
L_1 = \{ \vec{r}_1 + t\vec{v}_1 \mid t \in \mathbb{R} \} \\
L_2 = \{ \vec{r}_2 + t\vec{v}_2 \mid t \in \mathbb{R} \}
\]

\[
\cos(\theta) = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\|\|\vec{v}_2\|}
\]

(Note: Angle should be between 0 and \(\pi/2\).)

**Example**
Do the lines from the previous example intersect? What's the angle?
Lecture 4: Lines and planes in space

Cross products

Lines in the plane

Lines in space (parametric)

Planes in space (implicit)
For planes in space, most convenient is a generalization of implicit rep. of lines.

**Principle**
Set of vectors perpendicular to given one is a plane (through $\vec{0}$).

Fix vector $\vec{n}$. Plane is

$$\{\vec{r} \mid \vec{n} \cdot \vec{r} = 0\} = \{(x, y, z) \mid ax + by + cz = 0\}$$  \hspace{1cm} (Passing through $\vec{0}$)

$$\{\vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0\} = \{(x, y, z) \mid ax + by + cz = s\}$$  \hspace{1cm} (General)

Main problem is often to find the normal vector.

**Question**
What is the plane through $P = (1, 1, 0)$, $Q = (0, 1, 1)$, and $R = (1, 0, 1)$?

**Answer**
The vectors $\vec{PQ} = (-1, 0, 1)$ and $\vec{PR} = (0, -1, 1)$ lie on the plane.

Normal vector is perpendicular to both: $\vec{n} = \vec{PQ} \times \vec{PR} = (1, 1, 1)$.

Plane is $\{\vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0\} = \{\vec{r} \mid \vec{n} \cdot \vec{r} = 2\} = \{(x, y, z) \mid x + y + z = 2\}$. 

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\]
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Planes in space: Angles

Can find angle between lines in $\mathbb{R}^2$ by angle between dir. vectors or between normal vectors.

\[
\begin{align*}
\{ \vec{r}_1 + t\vec{v}_1 \mid t \in \mathbb{R} \} & \quad \{ \vec{r}_2 + t\vec{v}_2 \mid t \in \mathbb{R} \} \\
\text{Direction vector } \vec{v}_1 &= (a_1, b_1) & \text{Direction vector } \vec{v}_2 &= (a_2, b_2) \\
\text{Normal vector } \vec{n}_1 &= (-b_1, a_1) & \text{Normal vector } \vec{n}_2 &= (-b_2, a_2) \\
\cos(\theta) &= \frac{\vec{v}_1 \cdot \vec{v}_2}{\| \vec{v}_1 \| \| \vec{v}_2 \|} = \frac{\vec{n}_1 \cdot \vec{n}_2}{\| \vec{n}_1 \| \| \vec{n}_2 \|}
\end{align*}
\]

For angles between planes in $\mathbb{R}^3$, use angle between normal vectors.

\[
\begin{align*}
\{ \vec{r} \in \mathbb{R}^3 \mid \vec{n}_1 \cdot \vec{r} = \vec{n} \cdot \vec{r}_1 \} & \quad \{ \vec{r} \in \mathbb{R}^3 \mid \vec{n}_2 \cdot \vec{r} = \vec{n} \cdot \vec{r}_2 \} \\
\text{Normal vector } \vec{n}_1 & \quad \text{Normal vector } \vec{n}_2 \\
\cos(\theta) &= \frac{\vec{n}_1 \cdot \vec{n}_2}{\| \vec{n}_1 \| \| \vec{n}_2 \|}
\end{align*}
\]

- Parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$ ($\vec{n}_2 = a\vec{n}_1$)
- Perpendicular if $\vec{n}_1$ is perpendicular to $\vec{n}_2$ ($\vec{n}_1 \cdot \vec{n}_2 = 0$)
Planes in space: Angles, cont.

For angle between **line** and **plane** in $\mathbb{R}^3$, look at angle between

- direction vector $\vec{v}_L$ of line and
- normal vector $\vec{n}_P$ of plane.

- **Perpendicular** if $\vec{v}_L$ is parallel to $\vec{n}_P$
- **Parallel** if $\vec{v}_L$ is perpendicular to $\vec{n}_P$
- In general,

$$\cos(\theta + \pi/2) = \frac{\vec{v}_L \cdot \vec{n}_P}{\| \vec{v}_L \| \| \vec{n}_P \|}.$$
Planes in space: Angles, cont.

For angle between line and plane in $\mathbb{R}^3$, look at angle between

- direction vector $\vec{v}_L$ of line and
- normal vector $\vec{n}_P$ of plane.

- *Perpendicular* if $\vec{v}_L$ is parallel to $\vec{n}_P$
- *Parallel* if $\vec{v}_L$ is perpendicular to $\vec{n}_P$
- In general,

$$\cos(\theta + \pi/2) = \frac{\vec{v}_L \cdot \vec{n}_P}{\|\vec{v}_L\|\|\vec{n}_P\|}.$$
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- In general,

$$\cos(\theta + \pi/2) = \frac{\vec{v}_L \cdot \vec{n}_P}{\|\vec{v}_L\|\|\vec{n}_P\|}.$$
Distance to lines and planes

For a **line** in $\mathbb{R}^2$, distance is given by dot product with normal:

$$L = \{ \vec{r} \in \mathbb{R}^2 \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}$$

$$\text{dist}(\vec{p}, L) = \text{comp}_\vec{n}(\vec{p} - \vec{r}_0) = \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}$$

For a **plane** in $\mathbb{R}^3$, distance is given by dot product with normal:

$$P = \{ \vec{r} \mid \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \}$$

$$\text{dist}(\vec{p}, P) = \text{comp}_\vec{n}(\vec{p} - \vec{r}_0) = \frac{\vec{n} \cdot (\vec{p} - \vec{r}_0)}{\|\vec{n}\|}$$

In both cases, sometimes easier to look at *unit normal vector* $\frac{\vec{n}}{\|\vec{n}\|}$. 
Announcements

- HW1 has been graded. Pick up in drop box, location TBA.
- HW1 solutions on CourseWorks (under “Class Files”).
- Homework due on Tuesday.
- No late homeworks. 10% penalty for homeworks more than 5 minutes late in class. (You may also put them in drop box on 4th floor of Math building by 12:30.)
- No office hours on Monday.
- Printed copies of slides are distributing.