Lecture 3: Determinants and cross product

http://www.math.columbia.edu/~dpt/F10/CalcIII/
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Projections: Geometric description

The dot product of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$= \|\vec{v}\| \|\vec{w}\| \cos(\theta).$$

The projection of $\vec{w}$ onto $\vec{v}$, written $\text{proj}_\vec{v}(\vec{w})$, is the closest point to $\vec{w}$ on the line containing $\vec{v}$.

Projections and dot product

$\vec{w} - \text{proj}_\vec{v}(\vec{w})$ is the vector in dir. of $\vec{v}$ so $\vec{w} - \text{proj}_\vec{v}(\vec{w})$ is perp. to $\vec{v}$.

$\text{proj}_\vec{v}(\vec{w}) = a\vec{v}$

(a to be determined)

Solve for $a$:

Theorem

$$\text{proj}_\vec{v}(\vec{w}) = \left( \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}.$$ 

(Note: Scalar times a vector.)

Theorem

$$\text{proj}_\vec{v}(\vec{w}) = \left( \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}.$$ 

The (signed) length of the projection is

$$\text{comp}_\vec{v}(\vec{w}) = \pm \|\text{proj}_\vec{v}(\vec{w})\| = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|}$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \text{comp}_\vec{w}(\vec{w})$$

$$= \text{comp}_{\vec{w}}(\vec{v}) \|\vec{w}\|$$
Matrices and determinants

Before getting to the cross product, we’ll look at determinants. Determinants are really important in linear algebra. In particular, crucial for understanding volume. We’ll see just a tiny taste. (This may be review.)

Definition
A matrix is a rectangular array of numbers.

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
\end{pmatrix}
\]

The determinant takes a square matrix and gives a number.

\[
\det(a) = a \\
\det\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} = ad - bc
\]

(This may be review.)

Determinants: Examples

\[
\det(a) = a \\
\det\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} = ad - bc
\]

Examples

\[
\det\begin{pmatrix}
1 & 2 \\
3 & 4 \\
\end{pmatrix} = (1)(4) - (2)(3) = -2 \\
\det\begin{pmatrix}
1 & 2 \\
1 & 2 \\
\end{pmatrix} = (1)(2) - (1)(2) = 0
\]

In general, if two rows of the matrix are the same, determinant is 0.

Determinants and area

Theorem
If \(\vec{v} = (v_1, v_2)\) and \(\vec{w} = (w_1, w_2)\) are two vectors in \(\mathbb{R}^2\), then

\[
\det\begin{pmatrix}
v_1 & v_2 \\
w_1 & w_2 \\
\end{pmatrix} = |v_1w_2 - v_2w_1|
\]

is the area of the parallelogram with sides \(\vec{v}\) and \(\vec{w}\).

Determinants and area: Proof

Theorem
\(\vec{v} = (v_1, v_2)\), \(\vec{w} = (w_1, w_2)\)

\[\det\begin{pmatrix}
v_1 & v_2 \\
w_1 & w_2 \\
\end{pmatrix}\]

is area of parallelogram.

Proof.

Area = (Base)(Height)

\[ = (\|\vec{v}\|)(\|\vec{w}\|\sin(\theta))\].

The vector \(\vec{v'} = (-v_2, v_1)\) is same length as \(\vec{v}\), perpendicular to \(\vec{v}\).

Area = \(\|\vec{v'}\|\|\vec{w}\|\cos(\theta')\|

\[ = |\vec{v'} \cdot \vec{w}| \\
\[ = |(-v_2)w_1 + (v_1)w_2|.
\]

Summary: Area = \(\|\vec{v}\|(|\text{comp}_{\vec{v'}} \vec{w}|) = |\vec{v'} \cdot \vec{w}|.\)
The determinant of a $3 \times 3$ matrix is
\[
\begin{vmatrix}
 u_1 & u_2 & u_3 \\
 v_1 & v_2 & v_3 \\
 w_1 & w_2 & w_3
\end{vmatrix}
= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)
\]
Write this as $\det(\vec{u}, \vec{v}, \vec{w})$, where $\vec{u}, \vec{v}, \vec{w}$ are the three rows.

This generalizes! (But we won’t use generalization.)

Exercise
Find the volumes of parallelepiped with given sides using $\det$.

Sign?

$\vec{i}, \vec{j}, \vec{k}$

$2\vec{j}, 3\vec{i}, \vec{k}$

$\vec{i} + \vec{j}, \vec{j}, \vec{k} - \vec{j}$

Cross product: Algebraic definition

The cross product takes two vectors in $\mathbb{R}^3$ and gives another vector in $\mathbb{R}^3$.

\[
(v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)
\]

Recall:
\[
\det(\vec{u}, \vec{v}, \vec{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)
\]

Therefore,
\[
\vec{u} \cdot (\vec{v} \times \vec{w}) = \det(\vec{u}, \vec{v}, \vec{w})
\]

Warning
Cross product $\vec{v} \times \vec{w}$ only works in $\mathbb{R}^3$!

Cross product: Geometric meaning

\[
(v_1, v_2, v_3) \times (w_1, w_2, w_3) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)
\]

Theorem
If $\vec{v}$ and $\vec{w}$ form an angle $\theta$ (with $\theta > 0$), then $\vec{v} \times \vec{w}$ is
1. perpendicular to $\vec{v}$ and $\vec{w}$,
2. has length $||\vec{v}|| ||\vec{w}|| \sin(\theta)$,
3. in direction given by right hand rule.

Right hand rule: If (right hand) fingers curl from $\vec{v}$ to $\vec{w}$, thumb points at $\vec{v} \times \vec{w}$. E.g., $\vec{i} \times \vec{j} = \vec{k}$.

Compare: $\vec{v} \cdot \vec{w} = ||\vec{v}|| ||\vec{w}|| \cos(\theta)$. 
Cross product: Examples

Two definitions of cross product:

\[ \vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \]

Example

\[ (0,1,0) \times (1,0,0) = \]
\[ (1,1,0) \times (0,1,0) = \]
\[ (1,0,1) \times (1,1,0) = \]

How to prove this? Could bash it out algebraically, as in the book. We'll prove it geometrically.

Properties of cross product

A product of two basis vectors is \( \pm \) the third basis vector.

\[ \hat{i} \times \hat{j} = \hat{k} \]
\[ \hat{j} \times \hat{i} = -\hat{k} \]
\[ \hat{k} \times \hat{j} = \hat{i} \]
\[ \hat{i} \times \hat{k} = -\hat{j} \]

In fact, this is enough to reconstruct formula for cross product:

\[ \vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \]

Cross product does not satisfy all the properties you expect:

\[ \vec{v} \times (\vec{w} + \vec{u}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u} \quad \text{(Distributive)} \]
\[ (a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w}) \]
\[ \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \neq \vec{w} \times \vec{v} \quad \text{(Anti-commutative)} \]
\[ \vec{v} \times (\vec{w} \times \vec{u}) \neq (\vec{v} \times \vec{w}) \times \vec{u} \quad \text{(Not associative)} \]

Anti-commutative means that the sign changes when you switch the two factors. You probably haven't seen this before! In particular,

\[ \vec{v} \times \vec{v} = 0. \]