Lecture 2: Dot product
Dylan Thurston

http://www.math.columbia.edu/~dpt/F10/CalcIII/

proj_{\vec{v}}(\vec{w})

September 9, 2010
Announcements

- Homework 1 is due Tuesday, at the beginning of class. If you must miss class, put the homework in the drop box (Math department, 4th floor) before 12:30. Explain your answers; write English.
- Give me feedback!
- Ask questions!
- Section 8 has a window.
- Next office hours: Monday, 2–3 PM.
Vectors reprise

Dot product introduction

Geometric meaning

Correlation

Projections
Vectors, reprise

A vector has a length and a direction. Often represented by an arrow; but starting point does not matter.

Operations

- **Scalar multiplication**: \( a\vec{v} \) is in direction of \( \vec{v} \), scaled by \( a \)
- **Vector addition**: Add components, or form a parallelogram
- **Vector subtraction**: Subtract components, or form a triangle
Vectors, reprise

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There are special vectors, pointing along the coordinate axes:

\[ \vec{i} = (1, 0) \quad \vec{i} = (1, 0, 0) \]
\[ \vec{j} = (0, 1) \quad \vec{j} = (0, 1, 0) \]
\[ \vec{k} = (0, 0, 1) \]

Any vector can be written as a combination of these (uniquely!):

\[ a\vec{i} + b\vec{j} + c\vec{k} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) \]
\[ = (a, 0, 0) + (0, b, 0) + (0, 0, c) \]
\[ = (a, b, c) \]

A set of vectors with this property is called a *basis*. You can think of a basis as a coordinate system.
Spherical coordinates

Represent a point \( p \in \mathbb{R}^3 \) by \((r, \theta, \phi)_{\text{sphere}}\) where

- \( r \) is the distance from the origin \( \vec{0} \) to \( p \)
- \( \theta \) is the same angle as in cylindrical coordinates, the angle around the \( z \)-axis, starting at the \( xz \)-plane
- \( \phi \) is angle from \( p \) to the \( z \)-axis

On Earth, \( r \) is altitude, \( \theta \) is longitude, \( \phi \) is related to latitude. Have \( 0 \leq r \), \( 0 \leq \theta < 2\pi \), \( 0 \leq \phi \leq \pi \).

Examples

- The vector \((1, 1, 0)_{\text{rect}}\) has spherical coordinates
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Converting spherical coordinates

Like for polar coordinates, it’s easy to convert *from* spherical coordinates:

\[
\begin{align*}
    z &= r \cos(\phi) \\
    x &= r \cos(\theta) \sin(\phi) \\
    y &= r \sin(\theta) \sin(\phi)
\end{align*}
\]

Sanity check: is \( r^2 = x^2 + y^2 + z^2 \)?

\[
\begin{align*}
    x^2 + y^2 + z^2 &= r^2(\cos^2(\theta) \sin^2(\phi) + \sin^2(\theta) \sin^2(\phi) + \cos^2(\phi)) \\
                      &= r^2(\sin^2(\phi) + \cos^2(\phi)) \\
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To convert *to* spherical coordinates, need to know how to compute angles.
Dot product does this!
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To convert to spherical coordinates, need to know how to compute angles. Dot product does this!
Lecture 2: Dot product

Vectors reprise

► Dot product introduction

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Definition

Is it possible to multiply two vectors in a useful way? The *dot product* gives one answer. We’ll define it in coordinates first.

Given two vectors \( \vec{v} = (v_1, \ldots, v_n) \) and \( \vec{w} = (w_1, \ldots, w_n) \). Their *dot product* is

\[
\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n
\]

Examples

\[
\begin{align*}
(1, 2) \cdot (3, 4) &= (1)(3) + (2)(4) = 11 \\
(-1, 0, 1) \cdot (2, 3, 4) &= (-1)(2) + (0)(3) + (1)(4) = 2 \\
(1, -1, 1, -1, 1) \cdot (2, 2, 2, 2, 2) &= 2 - 2 + 2 - 2 + 2 = 2 \\
(1, 2) \cdot (1, 2) &= \\
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(1, 2) \cdot (1, 2) & = 1^2 + 2^2 = 5 \\
(1, 2) \cdot (-2, 1) & = (1)(-2) + (2)(-1) = 0
\end{align*}
\]
Properties

Dot product: $(v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) = v_1 w_1 + \cdots + v_n w_n$

Remarks

▶ Don’t confuse dot product $\vec{v} \cdot \vec{w}$ with scalar multiplication $a\vec{v}$!
▶ Can’t take dot product of (e.g.) a vector in $\mathbb{R}^2$ with a vector in $\mathbb{R}^3$!

Dot product has properties you expect, as long as they make sense.

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$ (Commutative)

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$ (Distributive)

$$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$$ (Associativity, kind of)

Question

Is the dot product associative?
Properties

Dot product: \((v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) = v_1w_1 + \cdots + v_nw_n\)

Remarks

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\[
\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \quad \text{(Commutative)}
\]

\[
(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \quad \text{(Distributive)}
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(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w}) \quad \text{(Associativity, kind of)}
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Question

Is the dot product associative?
Vectors reprise

Dot product introduction

► Geometric meaning

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Lengths and cosines

Observation
\[ \vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\vec{v}\|^2. \]

Can we generalize this?

Theorem
\[ \vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta), \text{ where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{w}. \]

Example
The angle between the vectors (1, 1) and (2, 3) is

Corollary
- The angle between \( \vec{v} \) and \( \vec{w} \) is \( \arccos \left( \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \right) \)
- \( \vec{v} \) and \( \vec{w} \) are orthogonal (perpendicular) just when \( \vec{v} \cdot \vec{w} = 0 \).
Lengths and cosines

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Example
The angle between the vectors \((1, 1)\) and \((2, 3)\) is
\[ \arccos \left( \frac{5}{\sqrt{2}\sqrt{13}} \right) \approx \arccos(0.98) \approx 11.3^\circ \]

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&\quad \vec{v} \text{ and } \vec{w} \text{ are orthogonal (perpendicular) just when } \vec{v} \cdot \vec{w} = 0.
\end{align*} \]
Examples

- $\vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cos(\theta)$, where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$.
- The angle between $\vec{v}$ and $\vec{w}$ is $\arccos \left( \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \cdot ||\vec{w}||} \right)$.
- $\vec{v}$ and $\vec{w}$ are orthogonal (perpendicular) just when $\vec{v} \cdot \vec{w} = 0$.

Examples

- Compute the angle between $(3, 3)$ and $(0, 5)$ using the dot product.
- Suppose $\vec{v}$ and $\vec{w}$ are unit vectors and the angle between $\vec{v}$ and $\vec{w}$ is $\pi/3$. What is $\vec{v} \cdot \vec{w}$?

Example

The angle $\phi$ in spherical coordinates is the angle between the point $p$ and the $z$-axis:

$$\phi = \arccos \left( \frac{p \cdot (0, 0, 1)}{||p||} \right) = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$
Proof

Theorem
\( \vec{v} \cdot \vec{w} = \| \vec{v} \| \| \vec{w} \| \cos(\theta) \), where \( \theta \) is the angle between \( \vec{v} \) and \( \vec{w} \).

Plausibility: If \( \vec{v} \) and \( \vec{w} \) are unit vectors with \( \vec{v} \) along the \( x \)-axis,
\( \vec{v} \cdot \vec{w} = (1, 0) \cdot (\cos(\theta), \sin(\theta)) = \cos(\theta) \).

Proof.
Recall the law of cosines: In a triangle with sides \( a, b, c \),
\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]
where \( \theta \) is the angle between sides \( a \) and \( b \).
Apply to the triangle below and expand out the dot products.
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Geometric meaning

► Correlation

Projections
Temperature vs. CO₂

Correlation measures how two sets of measurements move together.

Temperature and CO₂ Records

Toy data

What does correlation mean?
Consider some toy, meaningless, far too small data.

<table>
<thead>
<tr>
<th>Year</th>
<th>CO$_2$ (ppm)</th>
<th>Avg. temp ($^\circ$C)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mauna Loa</td>
<td>Central Park, NYC</td>
</tr>
<tr>
<td>2004</td>
<td>377.5</td>
<td>12.47</td>
</tr>
<tr>
<td>2005</td>
<td>379.8</td>
<td>13.33</td>
</tr>
<tr>
<td>2006</td>
<td>381.9</td>
<td>13.41</td>
</tr>
<tr>
<td>2007</td>
<td>383.7</td>
<td>13.09</td>
</tr>
<tr>
<td>2008</td>
<td>385.6</td>
<td>12.93</td>
</tr>
<tr>
<td>2009</td>
<td>387.3</td>
<td>12.37</td>
</tr>
</tbody>
</table>

http://data.giss.nasa.gov/gistemp/station_data/

Write as two vectors:

\[
\vec{v}_{\text{co2}} = (377.5, 379.8, 381.9, 383.7, 385.6, 387.3)
\]

\[
\vec{v}_{\text{temp}} = (12.47, 13.33, 13.41, 13.09, 12.93, 12.37)
\]
Correlation as angle

$$\vec{v}_{co2} = (377.5, 379.8, 381.9, 383.7, 385.6, 387.3)$$
$$\vec{v}_{temp} = (12.47, 13.33, 13.41, 13.09, 12.93, 12.37)$$

Correlation is related to angle between these two vectors. Want result to be independent of origin (eg, °C vs °K) and units (eg, °C vs °F).

General procedure:

1. Recenter data so mean is 0.
2. Divide by norm to get unit vector.
3. Compute the dot product.

Steps 2 and 3 compute the angle (in a high-dimensional space).
Let’s compute!

\[ \vec{v}_{\text{co2}} = (377.5, 379.8, 381.9, 383.7, 385.6, 387.3) \]

\[ \vec{v}_{\text{temp}} = (12.47, 13.33, 13.41, 13.09, 12.93, 12.37) \]

Recenter:

\[ \text{mean}_{\text{co2}} = 382.6 \]

\[ \text{mean}_{\text{temp}} = 12.93 \]

\[ \vec{v}_{\text{co2,cent}} = (-5.13, -2.83, -0.73, 1.07, 2.97, 4.67) \]

\[ \vec{v}_{\text{temp,cent}} = (-0.46, 0.40, 0.48, 0.16, 0.00, -0.56) \]

Normalize:

\[ \| \vec{v}_{\text{co2,cent}} \| = 8.16 \]

\[ \| \vec{v}_{\text{temp,cent}} \| = 0.97 \]

\[ \vec{v}_{\text{co2,norm}} = (-0.63, -0.35, -0.09, 0.13, 0.36, 0.57) \]

\[ \vec{v}_{\text{temp,norm}} = (-0.47, 0.41, 0.49, 0.16, 0.00, -0.58) \]

Dot product:

\[ \vec{v}_{\text{co2,norm}} \cdot \vec{v}_{\text{temp,norm}} = -0.20 \]
Let’s compute!

\[ \vec{v}_{\text{co2}} = (377.5, 379.8, 381.9, 383.7, 385.6, 387.3) \]
\[ \vec{v}_{\text{temp}} = (12.47, 13.33, 13.41, 13.09, 12.93, 12.37) \]

Recenter:

\[ \text{mean}_{\text{co2}} = 382.6 \]
\[ \text{mean}_{\text{temp}} = 12.93 \]
\[ \vec{v}_{\text{co2,cent}} = (-5.13, -2.83, -0.73, 1.07, 2.97, 4.67) \]
\[ \vec{v}_{\text{temp,cent}} = (-0.46, 0.40, 0.48, 0.16, 0.00, -0.56) \]

Normalize:

\[ \| \vec{v}_{\text{co2,cent}} \| = 8.16 \]
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Summary

\[ \vec{v}_{\text{co2}} = (377.5, 379.8, 381.9, 383.7, 385.6, 387.3) \]
\[ \vec{v}_{\text{temp}} = (12.47, 13.33, 13.41, 13.09, 12.93, 12.37) \]
\[ \ldots \]
\[ \vec{v}_{\text{co2,norm}} \cdot \vec{v}_{\text{temp,norm}} = -0.20 \]

Correlation is weak and negative for this data: as CO$_2$ increases, temperature tends to decrease.
There is a correlation between CO$_2$ and surface temperature, but there's a lot of noise. We didn't take anywhere near enough data.

Exercise
Is there a correlation between CO$_2$ and the year?

Warning
Correlation is not causation!
**General procedure**

To compute the correlation between data vectors $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n)$:

1. Compute mean($\vec{v}$) = $(v_1 + \cdots + v_n)/n$
2. Compute *recentered* vector $\vec{v}_{\text{cent}} = \vec{v} - \text{mean(}\vec{v})(1, \ldots, 1)$
3. Compute *renormalized* vector $\vec{v}_{\text{norm}} = \vec{v}_{\text{cent}}/\|\vec{v}_{\text{cent}}\|$
4. Do same for $\vec{w}$
5. *Correlation* is $\vec{v}_{\text{norm}} \cdot \vec{w}_{\text{norm}}$

**Exercise**

What is the correlation between $(1, 2, 3)$ and $(1, 4, 1)$?
Vectors reprise

Dot product introduction

Geometric meaning

Correlation

▶ Projections
The *projection* of $\vec{w}$ onto $\vec{v}$, written $\text{proj}_{\vec{v}}(\vec{w})$, is the closest point to $\vec{w}$ on the line containing $\vec{v}$. 

![Diagram of vector projection](image-url)
The *projection* of $\vec{w}$ onto $\vec{v}$, written $\text{proj}_\vec{v}(\vec{w})$, is the closest point to $\vec{w}$ on the line containing $\vec{v}$.
The *projection* of \( \vec{w} \) onto \( \vec{v} \), written \( \text{proj}_{\vec{v}}(\vec{w}) \), is the closest point to \( \vec{w} \) on the line containing \( \vec{v} \).
Algebraic description

proj\(_{\vec{v}}\)(\vec{w}) is the vector in dir. of \(\vec{v}\) so \(\vec{w} - \text{proj}_{\vec{v}}(\vec{w})\) is perp. to \(\vec{v}\).

\[
\text{proj}_{\vec{v}}(\vec{w}) = a\vec{v} \quad (a \text{ to be determined})
\]

\[
(\vec{w} - \text{proj}_{\vec{v}}(\vec{w})) \cdot \vec{v} = (\vec{w} - a\vec{v}) \cdot \vec{v} = 0
\]

\[
\vec{w} \cdot \vec{v} = a\vec{v} \cdot \vec{v}
\]

\[
a = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}
\]
Algebraic description

\[ \text{proj}_\mathbf{v}(\mathbf{w}) \text{ is the vector in dir. of } \mathbf{v} \text{ so } \mathbf{w} - \text{proj}_\mathbf{v}(\mathbf{w}) \text{ is perp. to } \mathbf{v}. \]

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\[ \text{proj}_\mathbf{v}(\mathbf{w}) = a\mathbf{v} \quad (a \text{ to be determined}) \]

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Algebraic description

proj_{\vec{v}}(\vec{w}) is the vector in dir. of \vec{v} so \vec{w} - proj_{\vec{v}}(\vec{w}) is perp. to \vec{v}.

proj_{\vec{v}}(\vec{w}) = a \vec{v} \quad (a \text{ to be determined})

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\vec{w} \cdot \vec{v} = a \vec{v} \cdot \vec{v}

a = \frac{\vec{w} \cdot \vec{v}}{||\vec{v}||^2}
Algebraic description

$$\text{proj}_v(\vec{w})$$ is the vector in dir. of $\vec{v}$ so $\vec{w} - \text{proj}_v(\vec{w})$ is perp. to $\vec{v}$.

$$\text{proj}_v(\vec{w}) = a\vec{v} \quad (a \text{ to be determined})$$

$$((\vec{w} - \text{proj}_v(\vec{w})) \cdot \vec{v}) = ((\vec{w} - a\vec{v}) \cdot \vec{v}) = 0$$

$$\vec{w} \cdot \vec{v} = a\vec{v} \cdot \vec{v}$$

$$a = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}$$
Algebraic description

$\mathbf{w} - \text{proj}_v(\mathbf{w})$ is the vector in dir. of $\mathbf{v}$ so $\mathbf{w} - \text{proj}_v(\mathbf{w})$ is perp. to $\mathbf{v}$.

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Algebraic description

Theorem

\[ \text{proj}_{\vec{v}}(\vec{w}) = \left( \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}. \]

(Note: This is a scalar times a vector.)

The length of the projection is

\[ \|\text{proj}_{\vec{v}}(\vec{w})\| = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|} \]