1. (1.1) An operation consists of two steps, the first of which can be completed in \( n \) ways. If the first step is completed in the \( j \)th way, the second step can be completed in \( m_j \) ways.

**(a)** Draw a tree diagram to find a formula for the total number of ways in which the operation can be completed.

*Solution.* Consider the tree diagram:

\[ \text{Start} \]

\[ \begin{array}{c}
\text{W1} \\
1 \quad 2 \quad \cdots \quad m_1 \\
\text{W2} \\
1 \quad 2 \quad \cdots \quad m_2 \\
\text{Wn} \\
1 \quad 2 \quad \cdots \quad m_n
\end{array} \]

The operation can be done in

\[ \sum_{j=1}^{n} m_j \]

many ways. //

**(b)** A student can study 0, 1, 2, or 3 hours for a history test on any given day. Use the formula obtained from (a) to verify that there are 13 ways in which the student can study at most 4 hours for the test on two consecutive days.

*Solution.* Consider that

- if the student studies for 0 hours on the first day, the student can study for 0, 1, 2, or 3 hours on the second day.
- if the student studies for 1 hour on the first day, the student can study for 0, 1, 2, or 3 hours on the second day.
- if the student studies for 2 hours on the first day, the student can study for 0, 1, or 2 hours on the second day.
- if the student studies for 3 hours on the first day, the student can study for 0 or 1 hours on the second day.

So there are

\[ 4 + 4 + 3 + 2 = 13 \]

ways in which this student can study for no more than four hours on two consecutive days. //

2. (1.8) Find an expression for the number of ways in which \( r \) distinguishable balls can be distributed among \( n \) boxes and use it to find the number of ways in which three different books can be distributed among 12 students.
Solution. For each of the \( r \) objects, there are \( n \) boxes to choose from so we see that there are \( n \cdot n \cdot \ldots \cdot n = n^r \) ways in which we can put the \( r \) objects into \( n \) boxes. Hence, for 12 students and 3 books, we can give those 3 books to the 12 students in \( 12^3 = 1728 \) ways. //

3. Show that

\[
\binom{n}{r} = \frac{n}{n-r} \cdot \binom{n-1}{r}
\]

Solution. Through algebraic massaging,

\[
\frac{n}{n-r} \cdot \binom{n-1}{r} = \frac{n}{n-r} \cdot \frac{(n-1)!}{(n-r)! \cdot (r)!} = \frac{n \cdot (n-1)!}{(n-r) \cdot ((n-r)-1)! \cdot r!} = \frac{n!}{(n-r)! \cdot r!} = \binom{n}{r} \text{. //}
\]

4. (1.17) Show that

\[
\sum_{r=0}^{n} r \binom{n}{r} = n2^{n-1}.
\]

Solution. By the Binomial Theorem,

\[
(x + 1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.
\]

Differentiating with respect to \( x \) on both sides yields

\[
n(x + 1)^{n-1} = \sum_{r=1}^{n} r \binom{n}{r} x^{r-1}.
\]

In particular, for \( x = 1 \), we see that

\[
n2^{n-1} = \sum_{r=1}^{n} r \binom{n}{r} = \sum_{r=0}^{n} r \binom{n}{r} \text{. //}
\]

5. A YT video has 8 time slots for advertisements. In how many ways can the ads be scheduled if

(a) there are 8 different ads to be shown?

Solution. There are \( 8! = 40320 \) ways of scheduling the ads. //
(b) there are 4 different ads to be shown and each is to be shown twice (not necessarily consecutively)?

Solution. Let the ads be labeled \( \{A, B, C, D\} \). We can place two occurrences of ad \( A \) in eight slots in \( \binom{8}{2} \) ways, \( B \) in the remaining 6 slots in \( \binom{6}{2} \) ways, \( C \) in the remaining 4 slots in \( \binom{4}{2} \) ways, and \( D \) in the remaining 2 slots in \( \binom{2}{2} \) ways. Therefore, there are

\[
\binom{8}{2} \binom{6}{2} \binom{4}{2} \binom{2}{2} = 2520
\]

ways to arrange the four ads in the desired manner. //

Alternatively, note that we have 4 indistinguishable ads and we wish to use 2 of each. So we count

\[
\binom{8}{2,2,2,2} = 2520
\]

ways to arrange the four ads in the desired manner. //

6. In how many ways can a group of 7 people be seated in a circle for a classroom discussion?

Solution. They can be seated in a circle in \( 6! = 720 \) ways.

7. Mrs. Brown has bought 9 different toys as birthday presents for her triplets. How many ways can she distribute the presents giving 3 toys to each triplet?

Solution. Here, we count

\[
\binom{9}{3} \binom{6}{3} \binom{3}{3} = 1680
\]

ways. //

8. Ms. Jackson has 4 hardback books, 6 paperback books, and 7 magazines. How many ways can she choose two of each to take on a trip?

Solution. Ms. Jackson wants to choose two books of each so we count

\[
\binom{4}{2} \binom{6}{2} \binom{7}{2} = 1890
\]

ways. //
2 Graded Homework 2

1. The event “A or B, but not both” is \((A \cap B^c) \cup (A^c \cap B)\). Find a formula for the probability of this even in terms of \(P(A)\), \(P(B)\), and \(P(A \cap B)\).

Solution. Observe that

\[
P((A \cap B^c) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c \cap B) - P((A \cap B^c) \cap (A^c \cap B))
\]

\[
= P(A \cap B^c) + P(A^c \cap B)
\]

since

\[(A \cap B^c) \cap (A^c \cap B) = \emptyset.\]

Moreover,

\[A = (A \cap B) \cup (A \cap B^c) \implies P(A) = P(A \cap B) + P(A \cap B^c)\]

since \(A \cap B\) and \(A \cap B^c\) are mutually exclusive. Similarly,

\[B = (A \cap B) \cup (A^c \cap B) \implies P(B) = P(A \cap B) + P(A^c \cap B)\]

since \(A \cap B\) and \(A^c \cap B\) are mutually exclusive. Then we have that

- \(P(A \cap B^c) = P(A) - P(A \cap B)\)
- \(P(A^c \cap B) = P(B) - P(A \cap B)\)

Finally,

\[
P((A \cap B^c) \cup (A^c \cap B)) = P(A) + P(B) - 2 \cdot P(A \cap B). \quad \|
\]

2. If \(A\) and \(B\) are mutually exclusive events, \(P(A) = 0.46\), and \(P(B) = 0.39\), find

(a) \(P(A^c) = 0.54\)
(b) \(P(A \cap B) = 0\)
(c) \(P(A \cup B) = 0.85\)
(d) \(P(A^c \cap B) = 0.39\)

3. If \(A\) and \(B\) are events, \(P(A) = 0.58\), \(P(B) = 0.37\), and \(P(A \cap B) = 0.29\), find

(a) \(P(A \cup B) = 0.66\)
(b) \(P(A^c \cap B^c) = 0.34\)
(c) \(P(A \cap B^c) = 0.29\)
(d) \(P(A^c \cap B) = 0.08\)

1known as exclusive or
4. In a standard deck of 52 playing cards, there are 4 suits divided into 13 ranks, 3 of which are face cards. If four cards from the deck are dealt at random, what is the probability of getting two face cards and two non-face cards?

Solution. There are 12 face cards total and we can draw two of those in
\[ \binom{12}{2} = 66 \]
ways. There are 40 non-face cards and we can draw two of those in
\[ \binom{40}{2} = 780 \]
ways. So there are
\[ 66 \cdot 780 = 51480 \]
ways to get two face cards and two non-face cards. There are
\[ \binom{52}{4} = 270725 \]
total 4-card hands. Therefore, the probability of drawing two face cards and two non-face cards is
\[ \frac{51480}{270725} = \frac{792}{4165}. \]

5. For visitors to a farmers’ market, the probabilities of buying tomatoes, raspberries, and zucchini are 0.59, 0.57, and 0.55, respectively. The probability of buying

- both tomatoes and raspberries is 0.35,
- both tomatoes and zucchini is 0.37,
- both raspberries and zucchini is 0.32, and
- all three of these is 0.22.

Find the probability of buying

(a) none of these,

Solution. First, the probability of buying one of these is
\[ 0.59 + 0.57 + 0.55 - 0.35 - 0.37 - 0.32 + 0.22 = 0.89. \]
So the probability of buying none of these is 0.11. \]

(b) exactly one of these, and

Solution. The probability is
\[ 0.09 + 0.12 + 0.08 = 0.29. \]

(c) at least two of these.

Solution. The probability is
\[ 0.13 + 0.15 + 0.1 + 0.22 = 0.6. \]
Alternatively, it can be seen with
\[ 1 - (0.11 + 0.29) = 0.6. \]
3 Graded Homework 3

1. Show that if \( P(B|A) = P(B) \) and \( P(B) \neq 0 \), then \( P(A|B) = P(A) \).

   \textit{Proof.} Let’s consider two cases. First, suppose \( P(A) = 0 \). Then,
   \[
P(A|B) = \frac{P(A \cap B)}{P(B)} = 0 = P(A).
   \]

   Now, let’s entertain the scenario where \( P(A) \neq 0 \). In a similar fashion,
   \[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B|A)} = \frac{P(A \cap B)}{P(A) \cdot P(B)} = P(A).
   \]

   This finishes the proof. \( \square \)

2. Show that if events \( A \) and \( B \) are independent, then events \( A^c \) and \( B^c \) are independent.

   \textit{Proof.} First, notice that
   \[
   A^c \cap B^c = (A \cup B)^c
   \]

   and that
   \[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
   = P(A) + P(B) - P(A) \cdot (B)
   \]

   which implies that
   \[
P(A^c \cap B^c) = P((A \cup B)^c)
   = 1 - P(A \cup B)
   = 1 - [P(A) + P(B) - P(A) \cdot (B)]
   = 1 - P(A) - P(B) + P(A) \cdot P(B)
   = (1 - P(A)) \cdot (1 - P(B))
   = P(A^c) \cdot P(B^c),
   \]

   the desired conclusion. \( \square \)

3. If events \( A, B, \) and \( C \) are independent, show that events \( A \) and \( B \cup C \) are independent.
Proof. Let’s calculate

\[
P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) \\
= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\
= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A) \cdot P(B) \cdot P(C) \\
= P(A) \cdot [P(B) + P(C) - P(B) \cdot P(C)] \\
= P(A) \cdot [P(B) + P(C) - P(B \cap C)] \\
= P(A) \cdot P(B \cup C).
\]

The proof is finished.

4. Consider the following Venn diagram and use it to conclude that it is possible to have

\[
P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)
\]

even when \( A, B, \) and \( C \) are not independent events.

![Venn Diagram]

Solution. Let’s first collect the probabilities of \( A, B, \) and \( C \):

- \( P(A) = \frac{19}{72} + \frac{13}{144} + \frac{11}{144} + \frac{5}{72} = \frac{1}{2} \)
- \( P(B) = \frac{11}{72} + \frac{13}{144} + \frac{11}{144} + \frac{1}{72} = \frac{1}{3} \)
- \( P(C) = \frac{5}{72} + \frac{11}{144} + \frac{1}{72} + \frac{43}{144} = \frac{11}{24} \)

Then we observe that

\[
P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{11}{24} = \frac{11}{144} = P(A \cap B \cap C).
\]

Now, to see that \( A, B, \) and \( C \) are not independent, notice that

\[
P(A \cap C) = \frac{7}{48}
\]
but

\[ P(A) \cdot P(C) = \frac{11}{48}. \]

That is, \( P(A \cap C) \neq P(A) \cdot P(C) \). //

5. Each time Chris makes a free throw attempt, the probability that he makes the basket is 0.6. He has three attempts (which may be assumed to be independent events). Find the probability that he gets

(a) a hit followed by two misses.

*Solution.* The probability is \((0.6)(0.4)(0.4) = 0.096\). //

(b) a hit and two misses in any order.

*Solution.* There are \(\binom{3}{1} = 3\) ways to arrange a hit and two misses. Each of those events has a probability of \((0.6)(0.4)(0.4) = 0.096\). Therefore, the probability that Chris gets a hit and two misses is \(3 \cdot 0.096 = 0.288\). //

6. Each bag in a large box contains 25 tulip bulbs. Half of the bags contain 5 bulbs that will give yellow tulips and 20 bulbs that will give red tulips; one third of the bags contain 10 bulbs that will give yellow tulips and 15 bulbs that will give red tulips; the remaining sixth of the bags contain 15 bulbs that will give yellow tulips and 10 bulbs that will give red tulips. A bag is selected at random from the box, and one bulb is randomly selected from that bag and planted.

(a) Find the probability that the tulip that grows from the selected bulb is yellow.

*Solution.* Let \(E\) be the event that the selected bulb is yellow. Let \(A\) be the event of selecting one of the bags with 5 yellow tulip bulbs, \(B\) be the event of selecting one of the bags with 10 yellow tulip bulbs, and \(C\) be the event of selecting one of the bags with 15 yellow tulip buds. Then we calculate

\[
P(E) = P(E|A) \cdot P(A) + P(E|B) \cdot P(B) + P(E|C) \cdot P(C)
= \frac{5}{25} \cdot \frac{1}{2} + \frac{10}{25} \cdot \frac{1}{3} + \frac{15}{25} \cdot \frac{1}{6}
= \frac{1}{3}. //
\]

(b) If the tulip that grows is red, what is the probability that it came from a bag with 10 bulbs for yellow tulips and 15 bulbs for red tulips?

*Solution.* Now let \(R\) be the event that the selected bulb was for a red tulip. Using \(A, B, \text{ and } C\) from part (a), we are trying to find \(P(B|R)\). Consider the tree diagram:
From this, we see that

\[
\frac{2}{5} + \frac{1}{5} + \frac{1}{15} = \frac{2}{3}
\]

is the probability of getting a bulb for a red tulip. The event \( B \) contributes \( \frac{1}{5} \) so we see that

\[
P(B|R) = \frac{1}{5} \cdot \frac{3}{10} = \frac{3}{10}.
\]
4 Graded Homework 4

1. For what values of $k$ can
   \[ f(x) = (1 - k)k^x \]
   be a probability distribution function for a discrete random variable $X$ with range \{0, 1, 2, \ldots\}? Justify your answer.

   **Solution.** If $k < 0$, $f(x)$ would take on negative values so we need $k \geq 0$. Also, note
   that, for $0 < k < 1$,
   \[
   \sum_{x=0}^{\infty} (1 - k)k^x = (1 - k) \cdot \sum_{x=0}^{\infty} k^x = (1 - k) \cdot \frac{1}{1-k} = 1.
   \]
   If $k > 1$, then the series diverges. Thus, $0 \leq k < 1$. //

2. A coin is biased so that it three times as likely to come up heads as tails. The coin is
tossed 4 times and the random variable $X$ is the number of heads.

   **(a)** Find the probability distribution of $X$.

   **Solution.** We present it as a table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(\frac{1}{4})^4 = \frac{1}{256}$</td>
</tr>
<tr>
<td>1</td>
<td>$4 \cdot (\frac{3}{4}) \cdot (\frac{1}{4})^3 = \frac{3}{64}$</td>
</tr>
<tr>
<td>2</td>
<td>$6 \cdot (\frac{3}{4})^2 \cdot (\frac{1}{4})^2 = \frac{27}{128}$</td>
</tr>
<tr>
<td>3</td>
<td>$4 \cdot (\frac{3}{4})^3 \cdot (\frac{1}{4}) = \frac{27}{64}$</td>
</tr>
<tr>
<td>4</td>
<td>$(\frac{3}{4})^4 = \frac{81}{256}$</td>
</tr>
</tbody>
</table>

   **(b)** Draw the probability histogram.
(c) Find the probability of getting at most 2 heads.

*Solution.* We calculate the probability of getting at most 2 heads to be
\[
\frac{1}{256} + \frac{3}{64} + \frac{27}{128} = \frac{67}{256}. \quad //
\]

(d) Find the distribution function of $X$.

*Solution.* Observe that
\[
F(x) = \begin{cases} 
0, & x < 0; \\
\frac{1}{256}, & 0 \leq x < 1; \\
\frac{13}{256}, & 1 \leq x < 2; \\
\frac{67}{256}, & 2 \leq x < 3; \\
\frac{175}{256}, & 3 \leq x < 4; \\
1, & 4 \leq x
\end{cases}
\]

(e) Graph the distribution function.
(f) Use the distribution function to find \( P(1 \leq X \leq 3) \) and \( P(X > 3) \).

\textit{Solution.} Note that
\[
P(1 \leq X \leq 3) = \frac{175}{256} - \frac{1}{256} = \frac{87}{128}
\]
and that
\[
P(X > 3) = 1 - \frac{175}{256} = \frac{81}{256}.
\]

3. The probability density function of the random variable \( Z \) is given by
\[
f(z) = \begin{cases} 
kze^{-z^2}, & z > 0; \\ 
0, & z \leq 0.
\end{cases}
\]

(a) Find \( k \) and graph this probability density.

\textit{Solution.} By integrating, we see that
\[
\int_{-\infty}^{\infty} f(z) \, dz = \int_{0}^{\infty} kze^{-z^2} \, dz = \frac{k}{2}.
\]
So \( k = 2 \).

Now, the graph is

(b) Find the distribution function of \( Z \) and sketch its graph.

\textit{Solution.} For \( z \leq 0 \), \( F(z) = 0 \). For \( z > 0 \), we calculate
\[
F(z) = \int_{0}^{z} 2te^{-t^2} \, dt \\
= \int_{u=0}^{u=z^2} e^{-u} \, du \\
= 1 - e^{-z^2}.
\]

Now, the graph is
(c) Find $P(1 \leq Z \leq 2)$.
   \textit{Solution.} Observe that
   \[ P(1 \leq Z \leq 2) = F(2) - F(1) = 1 - e^{-2^2} - 1 + e^{-1^2} = \frac{e^3 - 1}{e^4} \approx 0.3496. \]

4. The distribution function of the random variable $Y$ is given by
   \[ F(y) = \begin{cases} 1 - (1 + y)e^{-y}, & y > 0; \\ 0, & y \leq 0. \end{cases} \]

(a) Find $P(Y \leq 1.5)$.
   \textit{Solution.} Here, we just need to calculate
   \[ F(1.5) = 1 - (1 + 1.5)e^{-1.5} \approx 0.442175. \]

(b) Find $P(2 < Y < 3)$.
   \textit{Solution.} Now,
   \[ F(3) - F(2) = 1 - (1 + 3)e^{-3} - 1 + (1 + 2)e^{-2} \approx 0.206858. \]

(c) Find $P(Y \geq 3.6)$.
   \textit{Solution.} Here,
   \[ 1 - F(3.6) = 1 - 1 + (1 + 3.6)e^{-3.6} \approx 0.125689. \]

(d) Find the probability density function of $Y$.
   \textit{Solution.} To get the density, we differentiate:
   \[ \frac{d}{dy} F(y) = -((1 + y)e^{-y} \cdot (-1) + e^{-y}) \]
   \[ = e^{-y} + ye^{-y} - e^{-y} \]
   \[ = ye^{-y}. \]

5. The values of $f(x, y)$, the joint probability distribution of $X$ and $Y$, are shown in the table below:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$ :</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/20</td>
<td>1/16</td>
<td>3/20</td>
<td>1/40</td>
</tr>
<tr>
<td>2</td>
<td>1/8</td>
<td>3/16</td>
<td>3/40</td>
<td>1/10</td>
</tr>
<tr>
<td>3</td>
<td>7/40</td>
<td>3/80</td>
<td>1/80</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Find $P(X = 0, 2 \leq Y \leq 3)$.
   \textit{Solution.} $P(X = 0, 2 \leq Y \leq 3) = \frac{3}{16} + \frac{3}{80} = \frac{9}{40}.$

(b) Find $P(X + Y \leq 2)$.
   \textit{Solution.} Observe that
   \[ P(X + Y \leq 2) = \frac{1}{20} + \frac{1}{16} + \frac{3}{20} + \frac{1}{8} + \frac{3}{16} + \frac{7}{40} = \frac{3}{4}. \]
(c) If \( F(x, y) \) is the joint distribution function of \( X \) and \( Y \), find \( F(1, 2) \).

**Solution.** The following is the joint distribution function:

\[
\begin{array}{c|cccc}
X & -1 & 0 & 1 & 2 \\
\hline
Y & 1 & 1/20 & 9/80 & 21/80 & 23/80 \\
& 2 & 7/40 & 17/40 & 13/20 & 31/40 \\
& 3 & 7/20 & 51/80 & 7/8 & 1 \\
\end{array}
\]

In particular, \( F(1, 2) = \frac{13}{20} \).

(d) Find the marginal distribution of \( X \).

\[
\begin{array}{c|cccc}
X & -1 & 0 & 1 & 2 \\
\hline
& 7/20 & 23/80 & 19/80 & 1/8 \\
\end{array}
\]

(e) Find the marginal distribution of \( Y \).

\[
\begin{array}{c|ccc}
Y & 1 & 2 & 3 \\
\hline
& 23/80 & 39/80 & 9/40 \\
\end{array}
\]
5 Graded Homework 5

1. The values of $f(x, y)$, the joint probability distribution of $X$ and $Y$, are shown in the table below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/20</td>
<td>1/16</td>
<td>3/20</td>
<td>1/40</td>
</tr>
<tr>
<td>2</td>
<td>1/8</td>
<td>3/16</td>
<td>3/40</td>
<td>1/10</td>
</tr>
<tr>
<td>3</td>
<td>7/40</td>
<td>3/80</td>
<td>1/80</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Find $P(X = 0, 2 \leq Y \leq 3)$.

*Solution.* $P(X = 0, 2 \leq Y \leq 3) = \frac{3}{16} + \frac{3}{80} = \frac{9}{40}$. //

(b) Find $P(X + Y \leq 2)$.

*Solution.* Observe that

$P(X + Y \leq 2) = \frac{1}{20} + \frac{1}{16} + \frac{3}{20} + \frac{1}{8} + \frac{3}{16} + \frac{7}{40} = \frac{3}{4}$. //

(c) If $F(x, y)$ is the joint distribution function of $X$ and $Y$, find $F(1, 2)$.

*Solution.* The following is the joint distribution function:

<table>
<thead>
<tr>
<th>$X$ :</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/20</td>
<td>9/80</td>
<td>21/80</td>
<td>23/80</td>
</tr>
<tr>
<td>2</td>
<td>7/40</td>
<td>17/40</td>
<td>13/20</td>
<td>31/40</td>
</tr>
<tr>
<td>3</td>
<td>7/20</td>
<td>51/80</td>
<td>7/8</td>
<td>1</td>
</tr>
</tbody>
</table>

In particular, $F(1, 2) = \frac{13}{20}$. //

(d) Find the marginal distribution of $X$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7/20</td>
<td>23/80</td>
<td>19/80</td>
<td>1/8</td>
</tr>
</tbody>
</table>

(e) Find the marginal distribution of $Y$.

<table>
<thead>
<tr>
<th>$Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>23/80</td>
<td>29/80</td>
<td>9/40</td>
</tr>
</tbody>
</table>

(f) Find the conditional distribution of $X$ given $Y = 2$.

$P(X|Y = 2) = \begin{bmatrix} 10/39 & 5/13 & 2/13 & 8/39 \end{bmatrix}$

(g) Find the conditional distribution of $Y$ given $X = 0$.

$P(Y|X = 0) = \begin{bmatrix} 5/23 & 15/23 & 3/23 \end{bmatrix}$
2. The joint probability density of continuous random variables $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} e^{-x-y}, & x > 0, \ y > 0; \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the joint distribution function $F(x, y)$ of $X$ and $Y$.

**Solution.** For $x > 0$ and $y > 0$, we have

$$F(x, y) = \int_0^x \int_0^y e^{-t} e^{-s} \, ds \, dt = \int_0^x e^{-t} (e^{-y} - 1) \, dt = (e^{-x} - 1)(e^{-y} - 1).$$

(b) Use the joint distribution function to find $P(1 < X < 2, 0 < Y < 1)$.

**Solution.** Graphically, we are interested in

\[
\begin{array}{c}
\text{Consider}
\end{array}
\]

which provides

$$P(1 < X < 2, 0 < Y < 1) = F(2, 1) - F(1, 1) = (e^{-2} - 1)(e^{-1} - 1) - (e^{-1} - 1)(e^{-1} - 1) = \frac{1}{e} + \frac{1}{e^3} - \frac{2}{e^5} \approx 0.1469959.$$ 

3. Find the joint probability density of continuous random variables $X$ and $Y$ whose joint distribution function is given by

$$F(x, y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}), & x > 0, \ y > 0; \\ 0, & \text{otherwise} \end{cases}$$

**Solution.** First,

$$\frac{\partial}{\partial x} F(x, y) = 2xe^{-x^2} (1 - e^{-y^2}).$$

Then

$$\frac{\partial}{\partial y} 2xe^{-x^2} (1 - e^{-y^2}) = 4xye^{-x^2} e^{-y^2}.$$
Hence, the joint probability density is
\[
f(x, y) = \begin{cases} 
4xye^{-x^2 - y^2}, & x > 0, y > 0; \\
0, & \text{otherwise} 
\end{cases}
\]

4. The joint probability density of continuous random variables \(X\) and \(Y\) is given by
\[
f(x, y) = \begin{cases} 
24x(1 - x - y), & x > 0, y > 0, x + y < 1; \\
0, & \text{otherwise} 
\end{cases}
\]

(a) Find \(P(X + Y < 1/2)\).

*Solution.* Here,
\[
P(X + Y < 1/2) = \int_0^{1/2} \int_0^{\frac{1}{2} - x} 24x(1 - x - y) \, dy \, dx = \frac{5}{16}.
\]

(b) Find the marginal density of \(X\).

*Solution.* First,
\[
\int_0^{1-x} 24x(1 - x - y) \, dy = 12x^3 - 24x^2 + 12x
\]
so the marginal density of \(X\) is given by
\[
g(x) = \begin{cases} 
12x^3 - 24x^2 + 12x, & 0 < x < 1; \\
0, & \text{otherwise} 
\end{cases}
\]

(c) Find the marginal density of \(Y\).

*Solution.* First,
\[
\int_0^{1-y} 24x(1 - x - y) \, dx = 4 - 12y + 12y^2 - 4y^3
\]
so the marginal density of \(Y\) is given by
\[
h(y) = \begin{cases} 
4 - 12y + 12y^2 - 4y^3, & 0 < y < 1; \\
0, & \text{otherwise} 
\end{cases}
\]

(d) Find \(P(X < 1/2)\).

*Solution.* Using the marginal density of \(X\),
\[
P(X < 1/2) = \int_0^{1/2} 12x^3 - 24x^2 + 12x \, dx = \frac{11}{16}.
\]
(e) Find the conditional density of $X$ given $Y = 1/2$.

**Solution.** First, notice that

$$f(x, 1/2) \over h(1/2) = 48x \cdot \left(\frac{1}{2} - x\right)$$

which provides

$$f(x|1/2) = \begin{cases} 48x \cdot \left(\frac{1}{2} - x\right), & 0 < x < 1/2; \\ 0, & \text{otherwise} \end{cases}$$

(f) Given that $Y = 1/2$, find the probability that $X > 1/4$.

**Solution.** Using $f(x|1/2)$, we see that

$$P(X > 1/4|Y = 1/2) = \int_{1/4}^{1/2} 48x \cdot \left(\frac{1}{2} - x\right) \, dx = \frac{1}{2}.$$  

(g) Determine whether or not $X$ and $Y$ are independent.

**Solution.** Since $f(x, y) \neq g(x) \cdot h(y)$, $X$ and $Y$ are dependent.
6 Graded Homework 6

1. The continuous random variable $X$ has probability density

$$f(x) = \begin{cases} x, & 0 < x < 1; \\ 2 - x, & 1 \leq x < 2; \\ 0, & \text{otherwise} \end{cases}$$

Find the expected value of $X$, the expected value of $X^2$, and the variance of $X$.

**Solution.** The expected value of $X$ is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx = \int_{0}^{1} x^2 \, dx + \int_{1}^{2} x(2 - x) \, dx = 1.$$ 

The expected value of $X^2$ is

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx = \int_{0}^{1} x^3 \, dx + \int_{1}^{2} x^2(2 - x) \, dx = \frac{7}{6}.$$ 

Hence,

$$\text{Var}(X) = \frac{7}{6} - 1 = \frac{1}{6}.$$

2. Find $\mu_r$ and $\sigma^2$ for the random variable $X$ which has probability density

$$f(x) = \begin{cases} \frac{1}{\ln(3) \cdot x}, & 1 < x < 3; \\ 0, & \text{otherwise} \end{cases}$$

**Solution.** For any positive integer $r$,

$$E(X^r) = \int_{1}^{3} \frac{x^r}{\ln(3) \cdot x} \, dx = \int_{1}^{3} \frac{x^{r-1}}{\ln(3)} \, dx = \frac{x^r}{r \cdot \ln(3)} \bigg|_{1}^{3} = \frac{3^r - 1}{r \cdot \ln(3)}.$$ 

Then we calculate

$$\sigma^2 = \frac{3^2 - 1}{2 \cdot \ln(3)} - \left( \frac{3 - 1}{\ln(3)} \right)^2 = \frac{4}{\ln(3)} - \frac{4}{\ln^2(3)} \approx 0.326815.$$ 

3. The continuous random variable $X$ has mean $\mu$ and its probability density $f(x)$ satisfies $f(x) = 0$ for all $x < 0$. Show that, for any constant $k > 0$, $P(X \geq k) \leq \frac{\mu}{k}$. 


Proof. Note that
\[ P(X \geq k) = \int_k^{\infty} f(x) \, dx \]
and that
\[ \int_k^{\infty} x \cdot f(x) \, dx \geq \int_k^{\infty} k \cdot f(x) \, dx = k \cdot P(X \geq k). \]
It follows that
\[ k \cdot P(X \geq k) \leq \int_k^{\infty} x \cdot f(x) \, dx \leq \int_0^{\infty} x \cdot f(x) \, dx = \mu \]
since \( f(x) = 0 \) for all \( x < 0 \). Therefore,
\[ P(X \geq k) \leq \frac{\mu}{k}. \]
\[ \square \]
7 Graded Homework 7

1. Find the moment-generating function for the random variable $X$ with density given by

$$f(x) = \begin{cases} 
1, & 0 < x < 1; \\
0, & \text{otherwise}. 
\end{cases}$$

Use the moment-generating function to find $\mu_1'$, $\mu_2'$, and the variance $\sigma^2$.

**Solution.** Notice that

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx} dx = \frac{e^t - 1}{t}.$$

Then

$$M_X'(t) = \frac{te^t - e^t + 1}{t^2}$$

and

$$M_X''(t) = \frac{t^2 e^t - 2te^t + 2e^t - 2}{t^3}.$$

Notice that

$$\mu_1' = M_X'(0) = \lim_{t \to 0} \frac{te^t - e^t + 1}{t^2} = \lim_{t \to 0} \frac{e^t}{2} = \frac{1}{2}.$$

Also,

$$\mu_2' = M_X''(0) = \lim_{t \to 0} \frac{t^2 e^t - 2te^t + 2e^t - 2}{t^3} = \lim_{t \to 0} \frac{e^t}{3} = \frac{1}{3}.$$

Lastly,

$$\sigma^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

2. The moment-generating function of a random variable $X$ is

$$M_X(t) = e^{3t+8t^2}$$
and the random variable \( Y \) is given by \( Y = \frac{X - 3}{4} \). Find the moment-generating function of \( Y \) and use it to find the mean and variance of \( Y \).

**Solution.** Using a theorem,

\[
M_Y(t) = e^{-3t/4} \cdot e^{3(t/4) + 8(t/4)^2} = e^{t^2/2}
\]

Notice that

\[
M'_Y(t) = te^{t^2/2}
\]

and

\[
M''_Y(t) = t^2 e^{t^2/2} + e^{t^2/2}.
\]

With this,

\[
\mu_Y = M'_Y(0) = 0,
\]

\[
\mu'_2 = M''_Y(0) = 1
\]

and we see that

\[
\sigma^2_Y = 1. //
\]

3. A coin is weighted so that the probability of it coming up heads is 60%. The coin is tossed twice, \( X \) is the number of heads, and \( Y \) is the number of heads on the first toss. Find the covariance of \( X \) and \( Y \).

**Solution.** First, let’s write the joint probability distribution:

<table>
<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.16</td>
<td>0.24</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.16</td>
<td>0.48</td>
<td>0.36</td>
</tr>
</tbody>
</table>

Recall that \( \text{cov}(X, Y) = E(XY) - E(X)E(Y) \) and note that

\[
E(XY) = (1)(1)(0.24) + (1)(2)(0.36) = 0.96,
\]

\[
E(X) = (1)(0.48) + (2)(0.36) = 1.2,
\]

and

\[
E(Y) = 0.6.
\]

Therefore,

\[
\text{cov}(X, Y) = 0.96 - (1.2)(0.6) = 0.24 //
\]

4. The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \begin{cases} 
(2x + y)/4, & 0 < x < 1, \ 0 < y < 2; \\
0, & \text{otherwise.}
\end{cases}
\]
(a) Find \( \text{cov}(X,Y) \)

**Solution.** First, observe that

\[
E(XY) = \int_0^1 \int_0^2 xy \cdot \frac{2x+y}{4} \, dy \, dx = \frac{2}{3},
\]

\[
E(X) = \int_0^1 \int_0^2 x \cdot \frac{2x+y}{4} \, dy \, dx = \frac{7}{12},
\]

and

\[
E(Y) = \int_0^1 \int_0^2 y \cdot \frac{2x+y}{4} \, dy \, dx = \frac{7}{6}.
\]

Then

\[
\text{cov}(X,Y) = \frac{2}{3} - \frac{7}{12} \cdot \frac{7}{6} = -\frac{1}{72}. //
\]

(b) Find the variance of \( Z = 4X + 5Y \).

**Solution.** Note that

\[
\text{Var}(Z) = 16 \cdot \text{Var}(X) + 25 \cdot \text{Var}(Y) + 2 \cdot 4 \cdot 5 \cdot \text{cov}(X,Y).
\]

To find this we need

\[
\text{Var}(X) = E(X^2) - E(X)^2
\]

\[
= \int_0^1 \int_0^2 x^2 \cdot \frac{2x+y}{4} \, dy \, dx - \frac{49}{144}
\]

\[
= \frac{5}{12} - \frac{49}{144}
\]

\[
= \frac{11}{144}
\]

and

\[
\text{Var}(Y) = E(Y^2) - E(Y)^2
\]

\[
= \int_0^1 \int_0^2 y^2 \cdot \frac{2x+y}{4} \, dy \, dx - \frac{49}{36}
\]

\[
= \frac{5}{3} - \frac{49}{36}
\]

\[
= \frac{11}{36}
\]

Therefore,

\[
\text{Var}(Z) = 16 \cdot \frac{11}{144} + 25 \cdot \frac{11}{36} + 2 \cdot 4 \cdot 5 \cdot \frac{-1}{72} = \frac{299}{36}. //
\]

(c) Find the conditional mean and conditional variance of \( Y \) given that \( X = 1/4 \).
**Solution.** The marginal density of $X$ is given by

$$g(x) = \int_0^2 \frac{2x + y}{4} \, dy = \frac{2x + 1}{2}$$

which leads us to

$$g(1/4) = \frac{3}{4}.$$ 

The probability density of $Y$ given $X = 1/4$ is

$$h(y|1/4) = \begin{cases} \frac{(1 + 2y)}{6}, & 0 < y < 2; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the conditional mean of $Y$ given $X = 1/4$ is

$$\int_0^2 y \cdot \frac{1 + 2y}{6} \, dy = \frac{11}{9}.$$ 

For the conditional variance, first compute

$$\int_0^2 y^2 \cdot \frac{1 + 2y}{6} \, dy = \frac{16}{9}.$$ 

Then the conditional variance of $Y$ given $X = 1/4$ is

$$\frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}. \checkmark$$
8 Graded Homework 8

1. Consider the random variable $X$ with a discrete uniform distribution $f(x) = \frac{1}{n}$ for $x = 1, 2, \ldots, n$

(a) Show that the moment-generating function of $X$ is given by

$M_X(t) = \frac{e^t(1 - e^{nt})}{n(1 - e^t)}$

**Solution.** Note that

$M_X(t) = E(e^{tX}) = \sum_{x=1}^{n} \frac{e^{tx}}{n} = \frac{e^t + e^{2t} + e^{3t} + \cdots + e^{nt}}{n}$

Let

$S = e^t + e^{2t} + e^{3t} + \cdots + e^{nt}$

and notice that

$e^t \cdot S = e^{2t} + e^{3t} + e^{4t} + \cdots + e^{(n+1)t}$

which provides

$(1 - e^t)S = S - e^t \cdot S = e^t - e^{(n+1)t} = e^t(1 - e^{nt})$.

Hence,

$M_X(t) = \frac{S}{n} = \frac{e^t(1 - e^{nt})}{n(1 - e^t)}$

as desired. //

(b) Find the mean of this distribution using the fact that

$\mu = \lim_{t \to 0} M_X'(t)$

**Solution.** Note that

$M_X'(t) = \frac{n(1 - e^t)(-ne^{(n+1)t} + e^t(1 - e^{nt})) - e^t(1 - e^{nt})(-ne^t)}{n^2(1 - e^t)^2}$

$= \frac{(1 - e^t)(-ne^{(n+1)t} + e^t - e^{(n+1)t}) + e^{2t} - e^{(n+2)t}}{n(1 - e^t)^2}$

$= \frac{-ne^{(n+1)t} + e^t - e^{(n+1)t} + ne^{(n+2)t} - e^{2t} + e^{(n+2)t} + e^{2t} - e^{(n+2)t}}{n(1 - e^t)^2}$

$= \frac{-ne^{(n+1)t} + e^t - e^{(n+1)t} + ne^{(n+2)t}}{n(1 - e^t)^2}$
Then,

\[
\mu = \lim_{t \to 0} M'_X(t) = \lim_{t \to 0} \frac{-ne^{(n+1)t} + e^t - e^{(n+1)t} + ne^{(n+2)t}}{n(1 - e^t)^2} = \lim_{t \to 0} \frac{-(n + 1)ne^{(n+1)t} + e^t - (n + 1)e^{(n+1)t} + (n + 2)ne^{(n+2)t}}{2n(1 - e^t)(-e^t)} = \lim_{t \to 0} \frac{-(n + 1)^2ne^{(n+1)t} + e^t - (n + 1)^2e^{(n+1)t} + (n + 2)^2ne^{(n+2)t}}{2n[(1 - e^t)(-e^t) + (-e^t)^2]} = \frac{-(n + 1)^2n + 1 - (n + 1)^2 + (n + 2)^2n}{2n} = \frac{n^2 + n}{2n} = \frac{n + 1}{2}.
\]

2. Verify that for the binomial distribution

\[
b(k + 1; n, p) = \frac{p(n - k)}{(k + 1)(1 - p)} \cdot b(k; n, p)
\]

Then use this to calculate the values of the binomial distribution for \(n = 6\) and \(p = 1/3\).

**Solution.** Notice that

\[
\frac{p(n - k)}{(k + 1)(1 - p)} \cdot b(k; n, p) = \frac{p(n - k)}{(k + 1)(1 - p)} \cdot \binom{n}{k} p^k(1 - p)^{n-k} = \frac{p(n - k)}{(k + 1)(1 - p)} \cdot \frac{n!}{k!(n - k)!} \cdot p^k(1 - p)^{n-k} = \frac{n!}{(k + 1)!(n - k - 1)!} \cdot p^{k+1}(1 - p)^{n-k-1} = \binom{n}{k + 1} p^{k+1}(1 - p)^{n-(k+1)} = b(k + 1; n, p)
\]

Now, for the rest, note that

\[
b(0; 6, 1/3) = \left(\frac{2}{3}\right)^6 = \frac{64}{729}.
\]
It follows that

\[
\begin{align*}
    k = 0 : \quad b(0; 6, 1/3) &= \frac{(1/3)(6-0)}{(0+1)(2/3)} \cdot \frac{64}{729} = \frac{64}{243} \\
    k = 1 : \quad b(1+1; 6, 1/3) &= \frac{(1/3)(6-1)}{(1+1)(2/3)} \cdot \frac{64}{243} = \frac{80}{243} \\
    k = 2 : \quad b(2+1; 6, 1/3) &= \frac{(1/3)(6-2)}{(2+1)(2/3)} \cdot \frac{80}{243} = \frac{160}{243} \\
    k = 3 : \quad b(3+1; 6, 1/3) &= \frac{(1/3)(6-3)}{(3+1)(2/3)} \cdot \frac{160}{243} = \frac{20}{243} \\
    k = 4 : \quad b(4+1; 6, 1/3) &= \frac{(1/3)(6-4)}{(4+1)(2/3)} \cdot \frac{20}{243} = \frac{4}{243} \\
    k = 5 : \quad b(5+1; 6, 1/3) &= \frac{(1/3)(6-5)}{(5+1)(2/3)} \cdot \frac{4}{243} = \frac{1}{729} \\
    k = 6 : \quad b(6+1; 6, 1/3) &= \frac{(1/3)(6-6)}{(6+1)(2/3)} \cdot \frac{1}{729} \\
\end{align*}
\]

3. A multiple choice test has 12 questions, each with 4 possible answers. A student taking the test guesses the answer to each question randomly.

(a) How many questions can the student be expected to get right?

Solution. Let \( X \) be the number of questions the student gets right. Then \( X \) is a binomial random variable with \( n = 12 \) and \( p = 1/4 \). Hence,

\[
E(X) = 12/4 = 3. \quad //
\]

(b) What is the variance of the number of correct questions?

Solution. The variance is

\[
12 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{4}. \quad //
\]

(c) What is the probability that the student will get exactly 4 questions correct?

Solution.

\[
P(X = 4) = \binom{12}{4} \left( \frac{1}{4} \right)^4 \left( \frac{3}{4} \right)^8 \approx 0.193578 \quad //
\]

(d) What is the probability that the student will get at most 4 questions correct?

Solution.

\[
P(X \leq 4) = \sum_{k=0}^{4} P(X = k) = \sum_{k=0}^{4} \binom{12}{k} \left( \frac{1}{4} \right)^k \left( \frac{3}{4} \right)^{12-k} \approx 0.842356 \quad //
\]

(e) What is the probability that the student will get at least 4 questions correct?

Solution.

\[
P(X \geq 4) = \sum_{k=4}^{12} P(X = k) = \sum_{k=4}^{12} \binom{12}{k} \left( \frac{1}{4} \right)^k \left( \frac{3}{4} \right)^{12-k} \approx 0.351221 \quad //
\]

4. To reduce the standard deviation of the binomial distribution by one half, what change must be made in the number of trials?
Solution. Given a binomial distribution with $n$ and $p$, the standard deviation is given by

$$\sqrt{np(1-p)}.$$  

To reduce that by one half, we need to find $m$ so that

$$\sqrt{mp(1-p)} = \frac{\sqrt{np(1-p)}}{2}$$

which leads us to $m = \frac{n}{4}$. Therefore, to reduce the standard deviation of the binomial distribution by one half, we need to reduce the number of trials by one quarter. //

5. A study is being conducted in which two independent groups of 10 participants are enrolled in a 6-month training program. In each group, individuals drop out of the program before the six months with a probability of 20% (independently of the other participants in their group). Find the probability that at least 9 participants complete the six month program in one of the two groups, but not in both.

Solution. Notice that, in one group, the probability that at least 9 participants complete the program is

$$b(9; 10, 0.8) + b(10; 10, 0.8) \approx 0.37581$$

Now, let $X$ be the number of groups in which at least 9 participants complete the six month program. This is a binomial distribution where $n = 2$ and $p \approx 0.37581$ so we compute

$$P(X = 1) \approx b(1; 2, 0.37581) \approx 0.46915$$ //
9 Graded Homework 9

1. The random variable $X$ has a Poisson distribution with parameter $\lambda$. Find $E(X)$, $E[X(X - 1)]$, and use these to derive the formulas for the mean and variance.

Solution. The probability distribution is given by

$$f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, 3, \ldots$. Then

$$E(X) = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda \cdot \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \cdot \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda.$$

Next,

$$E(X(X - 1)) = \sum_{k=0}^{\infty} k(k - 1) \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=2}^{\infty} k(k - 1) \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-2)!}$$

$$= \lambda^2 \cdot \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^{k-2}}{(k-2)!}$$

$$= \lambda^2 \cdot \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \lambda^2.$$

Note that $E(X(X - 1)) = E(X^2) - E(X)$. So the mean is $\lambda$ and the variance is

$$E(X^2) - E(X)^2 = [E(X^2) - E(X)] + E(X) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$
2. The probability that a randomly selected person openly believes that the earth is flat is 12%. If people are selected at random and asked about their earthly opinions, what is the probability that the sixth person asked will be the second to believe that the earth is flat? Name the probability distribution that applies.

Solution. This is a negative binomial distribution.

The probability that the sixth person asked is the second to believe that the earth is flat is

\[
\binom{5}{1} (0.12)^2 (0.88)^4 \approx 0.043178
\]

3. Of thirty qualified applicants for an office job, seventeen live within ten miles of the office. If four applicants are chosen at random to be interviewed, what is the probability that exactly two of them live within ten miles of the office? Name the probability distribution that applies.

Solution. This is a hypergeometric distribution.

The probability that exactly two of four randomly chosen applicants live within ten miles of the office is

\[
\frac{\binom{17}{2} \binom{13}{2}}{\binom{30}{4}} = \frac{3536}{9135} \approx 0.387082649
\]

4. A math help webpage averages 15 hits per hour. What is the probability that, between 2pm and 2:30pm today, the website will have between 5 and 7 hits? Name the probability distribution that applies.

Solution. This is a Poisson distribution.

Since the average is 15 hits per hour, there are 15/2 = 7.5 per half-hour, on average. Now, we use

\[
\sum_{k=5}^{7} e^{-7.5} \cdot \frac{(7.5)^k}{k!} = e^{-7.5} \cdot \left[ \frac{(7.5)^5}{5!} + \frac{(7.5)^6}{6!} + \frac{(7.5)^7}{7!} \right] \approx 0.392577
\]

5. The number of cars passing an exit on a busy highway in a one-minute period is a random variable with a Poisson distribution with parameter \( \lambda = 10 \). Find the probability that the waiting time between two cars passing the exit is less than 4 seconds. Name the probability density function that applies.

Solution. This is an exponential distribution.

Since the rate is in terms of one-minute intervals, we translate 4 seconds to 4/60 = 1/15 minutes. Hence, the probability that the waiting time between two cars passing the exit is less than 4 seconds is

\[
\int_0^{1/15} 10 \cdot e^{-10x} \, dx \approx 0.48658288
\]
10 Graded Homework 10

1. Show that the normal distribution has an absolute maximum at \( x = \mu \) and inflection points at \( x = \mu \pm \sigma \).

Solution. Note that, for
\[
f(x) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma^2}
\]
\[
df \quad dx = -\frac{(x-\mu)}{\sigma^2} \cdot f(x)
\]
and that
\[
\frac{d^2f}{dx^2} = \frac{(x-\mu)^2}{\sigma^4} \cdot f(x) - \frac{1}{\sigma^2} \cdot f(x)
\]
\[
= f(x) \cdot \left[ \frac{(x-\mu)^2}{\sigma^5} - \frac{1}{\sigma^3} \right]
\]
\[
= \frac{f(x)}{\sigma^5} \cdot [(x-\mu)^2 - \sigma^2]
\]

For the maximum, we check
\[
f'(x) = 0
\]
which only happens for \( x = \mu \). Since \( f''(\mu) < 0 \), we see that \( f(x) \) has a local maximum at \( x = \mu \), though, since we know the shape of the graph, this local maximum is additionally a global maximum.

For the inflection points, we need
\[
f''(x) = 0
\]
which happens if and only if
\[
(x-\mu)^2 - \sigma^2 = 0 \iff (x-\mu)^2 = \sigma^2 \iff x - \mu = \pm \sigma \iff x = \mu \pm \sigma.
\]

To check that both of these are indeed inflection points, notice that

- \( f''(x) > 0 \) for \( x < \mu - \sigma \),
- \( f''(x) < 0 \) for \( \mu - \sigma < x < \mu + \sigma \), and
- \( f''(x) > 0 \) for \( x > \mu + \sigma \). //

2. Let \( X \) be a random variable having the standard normal distribution and let \( Y = X^2 \). Show that \( \text{cov}(X,Y) = 0 \) though \( X \) and \( Y \) are dependent.

Solution. One way to see that \( X \) and \( Y \) are dependent is the following. Note that \( P(Y < 1) < 1 \) but that \( P(Y < 1|1 < X < 1) = 1 \) which gives \( P(Y < 1) \neq P(Y < 1|1 < X < 1) \). Hence \( X \) and \( Y \) are dependent.
For the covariance, we will compute $E(XY) - E(X)E(Y)$. We can use computational software to compute

$$E(X \cdot X^2) = E(X^3) = \int_{-\infty}^{\infty} x^3 \cdot \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx = 0.$$  

Otherwise, let $w = -\frac{x^2}{2}$ and notice that $dw = -x \, dx$ which implies $dx = (-1/x) \, dw$. Also, $2w = -x^2$. Then

$$\int x^3 \cdot \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx = \int x^3 \cdot \frac{\exp(w)}{\sqrt{2\pi}} \cdot \frac{1}{x} \, dw$$

$$= \int -x^2 \cdot \frac{\exp(w)}{\sqrt{2\pi}} \, dw$$

$$= \int 2w \cdot \frac{\exp(w)}{\sqrt{2\pi}} \, dw$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int we^w \, dw.$$

Now, by integration by parts, we let

$$u = w \quad \frac{du}{dw} = 1 \quad dv = e^w \, dw \quad v = e^w$$

which provides

$$\int we^w \, dw = we^w - \int e^w \, dw = we^w - e^w + C.$$

Substituting back for $w$, we see that

$$\int x^3 \cdot \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx = -\sqrt{\frac{2}{\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \cdot \left[\frac{x^2}{2} + 1\right] + C$$

$$= -\frac{(x^2 + 1) \cdot \exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} + C.$$  

Then the fact that $E(X^3) = 0$ follows from the fact that the exponential factor asymptotically dominates the polynomial factor.

Since $X$ has the standard normal distribution, we already know that $E(X) = 0$. Hence,

$$\text{cov}(X, Y) = E(X^3) - E(X^2)E(x) = 0.$$  

3. The offspring of a distant alien life-form are so that their birthweight is a normally distributed random variable with a mean of 4200 grams and a standard deviation of 389 grams.
(a) What is the probability that an alien baby will weigh between 2550 grams and 4970 grams?

\textit{Solution.} First, we convert to the standard normal:

\[
\frac{2550 - 4200}{389} \approx -4.2416 \quad \frac{4970 - 4200}{389} \approx 1.9794
\]

and compute

\[
\int_{-4.2416}^{1.9794} \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx \approx 0.976103
\]

(b) What is the probability that an alien baby will weigh less than 3500 grams?

\textit{Solution.} We use

\[
\frac{3500 - 4200}{389} \approx -1.7995
\]

to compute

\[
\int_{-\infty}^{-1.7995} \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx \approx 0.0359698
\]

(c) The alien babies whose birthweight is in the top 5% are immediately taken for training and future government service. What birthweights will qualify?

\textit{Solution.} Using a \(z\)-table or WolframAlpha, we see that \(z \approx 1.645\) is so that

\[
\int_{z}^{\infty} \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} \, dx \approx 0.05
\]

The birthweights \(b\) that qualify then are found by

\[
1.645 = \frac{b - 4200}{389} \implies b \approx 4839.905
\]

Hence, any alien baby weighing over 4839.905 at birth will qualify for training. //