M118 Homework Problems

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1 Graded Homework 1

1. Let the universal set be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and
   - $X = \{1, 3, 5, 7, 9, 11\}$
   - $Y = \{2, 4, 5, 7, 8, 10\}$
   - $Z = \{1, 2, 3, 4, 5\}$

   List the elements of the sets
   (a) $(X \cup Y)' = \{6, 12\}$. //
   (b) $(Z' \cap Y) \cup X = \{1, 3, 5, 7, 8, 9, 10, 11\}$. //
   (c) $(X \cup Y') \cap Z = \{1, 3, 5\}$. //

2. Let the universal set $U$ be all people, $C$ be all people who drink coffee, $S$ be all people who drink soda, and $T$ be all people who drink tea.

   (a) Describe, in words, the set $(C \cap S)' \cap T$ using the above descriptions. Be specific for full credit.

      Solution. The set $(C \cap S)' \cap T$ is the set of all tea drinkers who don’t drink both coffee and soda. //

   (b) Describe, in words, the difference between $(C \cup S) \cap T'$ and $C \cup (S \cap T')$ or explain why they are the same. (Hint. It may help to draw a Venn diagram.)

      Solution. In the set $(C \cup S) \cap T'$ there can be no tea drinkers since $(C \cup S) \cap T' \subseteq T'$. If there is a tea drinker who also drinks coffee, then they would be in the set $C \cup (S \cap T')$ since $C \cap T \subseteq C \subseteq C \cup (S \cap T')$.

      So these sets are not the same. //

   To see the difference visually, consider the Venn diagrams:

   ![Venn diagram 1](image1)
   ![Venn diagram 2](image2)

   $(C \cup S) \cap T'$ is shaded
   $C \cup (S \cap T')$ is shaded

3. Let $U$ be a universal set containing two sets $A$ and $B$ so that $n(U) = n(A \cup B) = 24$, $n(A) = 15$, and $n(B) = 19$. Find $n(A \cap B')$. 

Solution. By the inclusion-exclusion principle,

\[
\frac{n(A \cup B) + n(A \cap B)}{24} + \frac{n(A \cap B)}{15} = \frac{n(A) + n(B)}{19} = 10.
\]

Then \(n(A \cap B') = n(A) - n(A \cap B) = 15 - 10 = 5. \//

Alternatively, we know that \(n(A \cup B) = 24\) and \(n(B) = 19\) so the number of things in only \(A\) and not \(B\) is

\[n(A \cap B') = 24 - 19 = 5. \//\]

4. Consider a universal set \(U\) which contains three sets \(A\), \(B\), and \(C\). Draw a Venn diagram and shade the regions corresponding to the following sets:

(a) \((A' \cup B)'\)

(b) \(A' \cup B \cup C\)
2 Homework 1 Replacement

1. Let the universal set $U$ be all students at Academico University, $F$ be the set of all freshmen, and $D$ be the set of all students that like donuts.

(a) Write in set-theoretic notation the set of

"all students that are freshmen and like donuts".

*Solution.* This is $F \cap D$. //

(b) Write in set-theoretic notation the set of

"all students that are freshmen and all students that like donuts".

*Solution.* This is $F \cup D$. //

Consider the statement “here we have a student which is a freshman and a student which likes donuts”. This could be interpreted as conveying the information that there are two students, one which is a freshman and the other which likes donuts. If we said something like “here we have a student which is a freshman and likes donuts”, we mean that we have one student which is both a freshman and likes donuts.

(c) Describe in words the set $F \cap D'$.

*Solution.* This set is “the set of all freshmen that don’t like donuts”. //

(d) Describe in words the set $F \cup D'$.

*Solution.* This set is “the set of all students which are freshmen or like donuts”. This contains the set of all freshmen and it contains all of the students which don’t like donuts. //

2. Let $U$ be the universal set containing two sets $A$ and $B$. Given that $n(U) = 60$, $n(A) = 33$, $n(B) = 27$, and $n(A \cap B) = 9$, find each of the following:

(a) $n(A \cup B) = 33 + 27 - 9 = 51$ by the inclusion-exclusion principle. //

(b) $n((A \cup B)') = 9$ by the Venn diagram below. //

(c) $n(A \cap B') = 24$ by the Venn diagram below. //

(d) $n(A' \cap B) = 18$ by the Venn diagram below. //

(e) $n((A \cap B') \cup (A' \cap B)) = 24 + 18 = 42$ by the Venn diagram below. //

(f) $n(((A \cap B') \cup (A' \cap B))') = 9 + 9 = 18$ by the Venn diagram below. //
3 Graded Homework 2

1. A marketing specialist surveyed 400 people to determine from what source they found out about a new phone app. The results were

<table>
<thead>
<tr>
<th></th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>From NameBook</td>
<td>180</td>
</tr>
<tr>
<td>From InstaPic</td>
<td>175</td>
</tr>
<tr>
<td>From YouFilm</td>
<td>165</td>
</tr>
<tr>
<td>From NameBook and InstaPic</td>
<td>80</td>
</tr>
<tr>
<td>From NameBook and YouFilm</td>
<td>90</td>
</tr>
<tr>
<td>From InstaPic and YouFilm</td>
<td>50</td>
</tr>
<tr>
<td>From all three sources</td>
<td>30</td>
</tr>
</tbody>
</table>

Let $N$, $I$, and $Y$ be the sets of those who found out from NameBook, InstaPic, and YouFilm, respectively. Fill in the regions of the Venn diagram below to answer (a) — (d).

(a) How many only found out through YouFilm? 55

(b) How many found out through NameBook and InstaPic, but not YouFilm? 50

(c) How many found out from only one source? $40 + 75 + 55 = 170$

(d) How many found out through at least two sources? $50 + 60 + 20 + 30 = 160$

2. A television talk show is to include 4 women and 3 men as panelists seated in a row of 7 chairs.

(a) In how many ways can the panelists be seated if the men and women are to be alternated?

Solution. 144 ways by

$$\begin{array}{ccccccc}
\text{W} & \cdot & \text{M} & \cdot & \text{W} & \cdot & \text{M} & \cdot & \text{W}
\end{array}$$
(b) In how many ways can the panelists be seated if the men must sit together and
the women must also sit together?

Solution. 288 ways by

\[
\begin{array}{cccc|ccc}
4 & 3 & 2 & 1 & 3 & 2 & 1 \\
W & W & W & W & M & M & M \\
3 & 2 & 1 & 4 & 3 & 2 & 1 \\
M & M & M & W & W & W & W \\
\end{array}
\]

Even more directly, notice that there are 2 ways to choose either women or men to
start with. Once we’ve made that choice, there are 4! ways to order the women and 3!
ways to order the men. Hence, there are

\[2 \cdot 4! \cdot 3! = 288\]

ways.

3. In the distant land of Thistopia, each inhabitant is labeled with two letters (of 26 total
letters) followed by three digits between 0 and 9. How many inhabitants can live in
Thistopia if every inhabitant must have a different label?

Solution. \(26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^3 = 676000.\)

4. How many five-digit codes can be made that begin with a 1, end with 0 or 5, and have
no repeated digits?

Solution. There is one choice for the first digit and 2 choices for the last digit. Once
those choice have been made, we’ve used two numbers. So there are

\[1 \cdot 8 \cdot 7 \cdot 6 \cdot 2 = 672\]

different five-digit codes that begin with a 1 and end with a 0 or 5.

5. An archivist has 5500 documents to store and wishes to index each document with a
code consisting of a case-sensitive letter followed by two digits. Is it possible to assign
a different code to each document? Explain your answer.

Solution. Since the letter is case-sensitive, there are \(26 \cdot 2 = 52\) choices for the first
slot. Then the number of different codes is \(52 \cdot 10 \cdot 10 = 5200.\) Since there are 5500
different documents to store and only 5200 different codes, there are not enough codes
to uniquely identify each document.
4 Graded Homework 3

1. There are two boxes on a table. Box A contains three balls numbered 1, 2, and 3. Box B contains four balls numbered 1, 2, 3, and 4. An experiment consists of selecting one ball from each box at random and observing their numbers.

(a) What is the sample space of this experiment?

**Solution.** There is not a unique way to represent the sample space but a good one is

\[ S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\} \]

where the first coordinate is for A and the second for B. //

(b) Write the event \( E = \text{“the sum of the numbers is 5”} \) as a subset of the sample space.

**Solution.** The event is \( \{(1,4), (2,3), (3,2)\} \). //

2. An experiment consists of rolling a pair of six-sided dice (one green and one red).

(a) What is the probability that the sum of the numbers is less than 4?

**Solution.** Recall that there are 36 pairs of rolls. The set of those that have a sum less than 4 are

\[ \{(1,1), (1,2), (2,1)\} \]

and so the probability is \( \frac{3}{36} = \frac{1}{12} \). //

(b) What is the probability that both dice have the same number?

**Solution.** The set of those rolls where both rolls have the same number is

\[ \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \]

so the probability is \( \frac{6}{36} = \frac{1}{6} \). //

(c) What is the probability that both dice have the same number or the sum is greater than 9?

**Solution.** The event \( A \) of those pairs with a sum greater than 9 is

\[ \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\} \]

so \( \Pr(B) = \frac{6}{36} = \frac{1}{6} \). The event \( B \) of the dice having the same number is

\[ \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \]

Then

\[ A \cap B = \{(5,5), (6,6)\} \]

and \( \Pr(A \cap B) = \frac{2}{36} = \frac{1}{18} \). By the Inclusion-Exclusion Principle,

\[ \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{18} = \frac{5}{18}. \]
Alternatively,

\[ A \cup B = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6), (1, 1), (2, 2), (3, 3), (4, 4)\} \]

so we see that

\[ \Pr(A \cup B) = \frac{10}{36} = \frac{5}{18}. \]

(d) What is the probability that the sum is greater than 12?

\textbf{Solution.} 0 since it is impossible.

3. A box contains colored balls in the following amounts:

<table>
<thead>
<tr>
<th>number</th>
<th>color</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>yellow</td>
</tr>
<tr>
<td>5</td>
<td>orange</td>
</tr>
<tr>
<td>7</td>
<td>red</td>
</tr>
<tr>
<td>4</td>
<td>purple</td>
</tr>
<tr>
<td>3</td>
<td>green</td>
</tr>
</tbody>
</table>

(a) If one ball is randomly selected from the box, what is the probability the ball drawn is green?

\textbf{Solution.} Notice that there are 8 + 5 + 7 + 4 + 3 = 27 balls in total. Then the probability of randomly selecting a green ball is \( \frac{3}{27} = \frac{1}{9}. \)

(b) If one ball is randomly selected from the box, what is the probability the ball drawn is red or orange?

\textbf{Solution.} There are 7 red balls and 5 orange balls and no balls that are both red and orange. So the probability of the randomly selected ball is red or orange is \( \frac{7 + 5}{27} = \frac{4}{9}. \)

4. At the School of Rhymes, the probability that a randomly selected student studies freestyling is 0.7, the probability that a randomly selected student studies beatboxing is 0.5, and the probability that a randomly selected student studies freestyling or beatboxing is 0.85. What is the probability that a randomly selected student studies freestyling but not beatboxing?

\textbf{Solution.} We can compute this with 0.85 − 0.5 = 0.35. That is, the probability that a randomly selected student studies freestyling but not beatboxing is 0.35.

5. Let \( A \) and \( B \) be events so that \( \Pr(A) = 0.6, \Pr(B) = 0.5, \) and \( \Pr(A' \cup B') = 0.6. \)

(a) Find \( \Pr(A \cap B). \)

\textbf{Solution.} Notice that, by De Morgan’s Laws, \( A' \cup B' = (A \cap B)' \). Hence,

\[ \Pr(A \cap B) = 1 - \Pr(A' \cup B') = 0.4. \]
(b) Find $\Pr(A \cup B')$.

*Solution.* By the Inclusion-Exclusion Principle,

$$\Pr(A \cup B') = \Pr(A) + \Pr(B') - \Pr(A \cap B').$$

By part (a) we see that $\Pr(A \cap B') = 0.6 - 0.4 = 0.2$. Also, $\Pr(B') = 1 - \Pr(B) = 1 - 0.5 = 0.5$. In the end,

$$\Pr(A \cup B') = 0.6 + 0.5 - 0.2 = 0.9.$$

6. An experiment consists of observing what a customer in a clothing store purchases. $E$ is the event “the customer buys a sweatshirt” and $F$ is the event “the customer buys a pair of jeans”. Are the events $E$ and $F$ mutually exclusive? Explain your answer.

*Solution.* No, $E$ and $F$ are not mutually exclusive because any customer can buy a sweatshirt and a pair of jeans.