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1 Graded Homework 1

1. Let the universal set be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and
   
   - $X = \{1, 3, 5, 7, 9, 11\}$
   - $Y = \{2, 4, 5, 7, 8, 10\}$
   - $Z = \{1, 2, 3, 4, 5\}$

   List the elements of the sets
   
   (a) $(X \cup Y)' = \{6, 12\}$. //
   (b) $(Z' \cap Y) \cup X = \{1, 3, 5, 7, 8, 9, 10, 11\}$. //
   (c) $(X \cup Y') \cap Z = \{1, 3, 5\}$. //

2. Let the universal set $U$ be all people, $C$ be all people who drink coffee, $S$ be all people who drink soda, and $T$ be all people who drink tea.

   (a) Describe, in words, the set $(C \cap S)' \cap T$ using the above descriptions. Be specific for full credit.

   **Solution.** The set $(C \cap S)' \cap T$ is the set of all tea drinkers who don’t drink both coffee and soda. //

   (b) Describe, in words, the difference between $(C \cup S) \cap T'$ and $C \cup (S \cap T')$ or explain why they are the same. (Hint. It may help to draw a Venn diagram.)

   **Solution.** In the set $(C \cup S) \cap T'$ there can be no tea drinkers since $(C \cup S) \cap T' \subseteq T'$. If there is a tea drinker who also drinks coffee, then they would be in the set $C \cup (S \cap T')$ since $C \cap T \subseteq C \subseteq C \cup (S \cap T')$.

   So these sets are not the same. //

   To see the difference visually, consider the Venn diagrams:

   ![Venn Diagrams]

   $(C \cup S) \cap T'$ is shaded
   
   $C \cup (S \cap T')$ is shaded

3. Let $U$ be a universal set containing two sets $A$ and $B$ so that $n(U) = n(A \cup B) = 24$, $n(A) = 15$, and $n(B) = 19$. Find $n(A \cap B')$. 

   ...
Solution. By the inclusion-exclusion principle,

\[
n(A \cup B) + n(A \cap B) = n(A) + n(B)
\]

\[
\frac{24 + n(A \cap B)}{n(A \cap B)} = 10.
\]

Then \(n(A \cap B') = n(A) - n(A \cap B) = 15 - 10 = 5.\) //

Alternatively, we know that \(n(A \cup B) = 24\) and \(n(B) = 19\) so the number of things in only \(A\) and not \(B\) is

\[
n(A \cap B') = 24 - 19 = 5.\) //

4. Consider a universal set \(U\) which contains three sets \(A, B,\) and \(C\). Draw a Venn diagram and shade the regions corresponding to the following sets:

(a) \((A' \cup B)'\)

(b) \(A' \cup B \cup C\)
2 Homework 1 Replacement

1. Let the universal set $U$ be all students at Academico University, $F$ be the set of all freshmen, and $D$ be the set of all students that like donuts.

(a) Write in set-theoretic notation the set of

“all students that are freshmen and like donuts”.

Solution. This is $F \cap D$. //

(b) Write in set-theoretic notation the set of

“all students that are freshmen and all students that like donuts”.

Solution. This is $F \cup D$. //

Consider the statement “here we have a student which is a freshman and a student which likes donuts”. This could be interpreted as conveying the information that there are two students, one which is a freshman and the other which likes donuts. If we said something like “here we have a student which is a freshman and likes donuts”, we mean that we have one student which is both a freshman and likes donuts.

(c) Describe in words the set $F \cap D'$.

Solution. This set is “the set of all freshmen that don’t like donuts”. //

(d) Describe in words the set $F \cup D'$.

Solution. This set is “the set of all students which are freshmen or like donuts”. This contains the set of all freshmen and it contains all of the students which don’t like donuts. //

2. Let $U$ be the universal set containing two sets $A$ and $B$. Given that $n(U) = 60$, $n(A) = 33$, $n(B) = 27$, and $n(A \cap B) = 9$, find each of the following:

(a) $n(A \cup B) = 33 + 27 - 9 = 51$ by the inclusion-exclusion principle. //

(b) $n((A \cup B)') = 9$ by the Venn diagram below. //

(c) $n(A \cap B') = 24$ by the Venn diagram below. //

(d) $n(A' \cap B) = 18$ by the Venn diagram below. //

(e) $n((A \cap B') \cup (A' \cap B)) = 24 + 18 = 42$ by the Venn diagram below. //

(f) $n(((A \cap B') \cup (A' \cap B))') = 9 + 9 = 18$ by the Venn diagram below. //
3 Graded Homework 2

1. A marketing specialist surveyed 400 people to determine from what source they found out about a new phone app. The results were

<table>
<thead>
<tr>
<th>Source Combination</th>
<th>Number of People</th>
</tr>
</thead>
<tbody>
<tr>
<td>NameBook</td>
<td>180</td>
</tr>
<tr>
<td>InstaPic</td>
<td>175</td>
</tr>
<tr>
<td>YouFilm</td>
<td>165</td>
</tr>
<tr>
<td>NameBook and InstaPic</td>
<td>80</td>
</tr>
<tr>
<td>NameBook and YouFilm</td>
<td>90</td>
</tr>
<tr>
<td>InstaPic and YouFilm</td>
<td>50</td>
</tr>
<tr>
<td>All three sources</td>
<td>30</td>
</tr>
</tbody>
</table>

Let $N$, $I$, and $Y$ be the sets of those who found out from NameBook, InstaPic, and YouFilm, respectively. Fill in the regions of the Venn diagram below to answer (a) — (d).

(a) How many only found out through YouFilm? 55
(b) How many found out through NameBook and InstaPic, but not YouFilm? 50
(c) How many found out from only one source? 40 + 75 + 55 = 170
(d) How many found out through at least two sources? 50 + 60 + 20 + 30 = 160

2. A television talk show is to include 4 women and 3 men as panelists seated in a row of 7 chairs.

(a) In how many ways can the panelists be seated if the men and women are to be alternated?

Solution. 144 ways by

\[
\begin{array}{ccccccc}
\text{W} & \cdot & \text{M} & \cdot & \text{W} & \cdot & \text{M} & \cdot & \text{W} \\
4 & \cdot & 3 & \cdot & 3 & \cdot & 2 & \cdot & 1 & \cdot & 1 \\
\end{array}
\]
(b) In how many ways can the panelists be seated if the men must sit together and the women must also sit together?

Solution. 288 ways by

\[
\begin{array}{ccccccc}
4 & 3 & 2 & 1 & 3 & 2 & 1 \\
W & W & W & W & M & M & M \\
3 & 2 & 1 & 4 & 3 & 2 & 1 \\
M & M & M & W & W & W & W
\end{array}
\]

Even more directly, notice that there are 2 ways to choose either women or men to start with. Once we’ve made that choice, there are 4! ways to order the women and 3! ways to order the men. Hence, there are

\[
2 \cdot 4! \cdot 3! = 288
\]

ways.

3. In the distant land of Thistopia, each inhabitant is labeled with two letters (of 26 total letters) followed by three digits between 0 and 9. How many inhabitants can live in Thistopia if every inhabitant must have a different label?

Solution. \(26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^3 = 676000\).

4. How many five-digit codes can be made that begin with a 1, end with 0 or 5, and have no repeated digits?

Solution. There is one choice for the first digit and 2 choices for the last digit. Once those choice have been made, we’ve used two numbers. So there are

\[
1 \cdot 8 \cdot 7 \cdot 6 \cdot 2 = 672
\]

different five-digit codes that begin with a 1 and end with a 0 or 5.

5. An archivist has 5500 documents to store and wishes to index each document with a code consisting of a case-sensitive letter followed by two digits. Is it possible to assign a different code to each document? Explain your answer.

Solution. Since the letter is case-sensitive, there are \(26 \cdot 2 = 52\) choices for the first slot. Then the number of different codes is \(52 \cdot 10 \cdot 10 = 5200\). Since there are 5500 different documents to store and only 5200 different codes, there are not enough codes to uniquely identify each document.
4 Graded Homework 3

1. There are two boxes on a table. Box A contains three balls numbered 1, 2, and 3. Box B contains four balls numbered 1, 2, 3, and 4. An experiment consists of selecting one ball from each box at random and observing their numbers.

(a) What is the sample space of this experiment?

\textit{Solution.} There is not a unique way to represent the sample space but a good one is

\[ S = \{ (1,1), (1,2), (1,3), (1,4), \\
(2,1), (2,2), (2,3), (2,4), \\
(3,1), (3,2), (3,3), (3,4) \} \]

where the first coordinate is for A and the second for B. //

(b) Write the event \( E = \text{“the sum of the numbers is 5”} \) as a subset of the sample space.

\textit{Solution.} The event is \( \{(1,4), (2,3), (3,2)\} \). //

2. An experiment consists of rolling a pair of six-sided dice (one green and one red).

(a) What is the probability that the sum of the numbers is less than 4?

\textit{Solution.} Recall that there are 36 pairs of rolls. The set of those that have a sum less than 4 are

\[ \{(1,1), (1,2), (2,1)\} \]

and so the probability is \( \frac{3}{36} = \frac{1}{12} \). //

(b) What is the probability that both dice have the same number?

\textit{Solution.} The set of those rolls where both rolls have the same number is

\[ \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \]

so the probability is \( \frac{6}{36} = \frac{1}{6} \). //

(c) What is the probability that both dice have the same number or the sum is greater than 9?

\textit{Solution.} The event \( A \) of those pairs with a sum greater than 9 is

\[ \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\} \]

so \( \Pr(B) = \frac{6}{36} = \frac{1}{6} \). The event \( B \) of the dice having the same number is

\[ \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \]

Then

\[ A \cap B = \{(5,5), (6,6)\} \]

and \( \Pr(A \cap B) = \frac{2}{36} = \frac{1}{18} \). By the Inclusion-Exclusion Principle,

\[ \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{18} = \frac{5}{18}. \]
Alternatively,

\[ A \cup B = \{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6), (1, 1), (2, 2), (3, 3), (4, 4)\} \]

so we see that

\[ \Pr(A \cup B) = \frac{10}{36} = \frac{5}{18}. \]

(d) What is the probability that the sum is greater than 12?

**Solution.** 0 since it is impossible.

3. A box contains colored balls in the following amounts:

<table>
<thead>
<tr>
<th>number</th>
<th>color</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>yellow</td>
</tr>
<tr>
<td>5</td>
<td>orange</td>
</tr>
<tr>
<td>7</td>
<td>red</td>
</tr>
<tr>
<td>4</td>
<td>purple</td>
</tr>
<tr>
<td>3</td>
<td>green</td>
</tr>
</tbody>
</table>

(a) If one ball is randomly selected from the box, what is the probability the ball drawn is green?

**Solution.** Notice that there are \(8 + 5 + 7 + 4 + 3 = 27\) balls in total. Then the probability of randomly selecting a green ball is \(\frac{3}{27} = \frac{1}{9}\).

(b) If one ball is randomly selected from the box, what is the probability the ball drawn is red or orange?

**Solution.** There are 7 red balls and 5 orange balls and no balls that are both red and orange. So the probability of the randomly selected ball is red or orange is \(\frac{7 + 5}{27} = \frac{4}{9}\).

4. At the School of Rhymes, the probability that a randomly selected student studies freestyling is 0.7, the probability that a randomly selected student studies beatboxing is 0.5, and the probability that a randomly selected student studies freestyling or beatboxing is 0.85. What is the probability that a randomly selected student studies freestyling but not beatboxing?

**Solution.** We can compute this with \(0.85 - 0.5 = 0.35\). That is, the probability that a randomly selected student studies freestyling but not beatboxing is 0.35.

5. Let \(A\) and \(B\) be events so that \(\Pr(A) = 0.6\), \(\Pr(B) = 0.5\), and \(\Pr(A' \cup B') = 0.6\).

(a) Find \(\Pr(A \cap B)\).

**Solution.** Notice that, by De Morgan’s Laws, \(A' \cup B' = (A \cap B)'\). Hence,

\[ \Pr(A \cap B) = 1 - \Pr(A' \cup B') = 0.4. \]
(b) Find $\Pr(A \cup B')$.

\textit{Solution.} By the Inclusion-Exclusion Principle,

$$\Pr(A \cup B') = \Pr(A) + \Pr(B') - \Pr(A \cap B').$$

By part (a) we see that $\Pr(A \cap B') = 0.6 - 0.4 = 0.2$. Also, $\Pr(B') = 1 - \Pr(B) = 1 - 0.5 = 0.5$. In the end,

$$\Pr(A \cup B') = 0.6 + 0.5 - 0.2 = 0.9. \quad \blacksquare$$

6. An experiment consists of observing what a customer in a clothing store purchases. $E$ is the event “the customer buys a sweatshirt” and $F$ is the event “the customer buys a pair of jeans”. Are the events $E$ and $F$ mutually exclusive? Explain your answer.

\textit{Solution.} No, $E$ and $F$ are not mutually exclusive because any customer can buy a sweatshirt and a pair of jeans. \quad \blacksquare
5 Homework 3 Extra Credit

1. There are two boxes on a table. Box A contains four balls numbered 1, 2, 3, and 4. Box B contains six balls numbered 1, 2, 3, 4, 5, and 6. An experiment consists of selecting one ball from each box at random and observing the numbers. What is the sample space of this experiment?

Solution. The sample space is

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \}.$$

2. At the Gastrodemia, the probability that a randomly selected student enjoys escargots is 0.75, the probability that a randomly selected student enjoys caviar is 0.54, and the probability that a randomly selected student enjoys escargots or caviar is 0.85. What is the probability that a randomly selected student enjoys escargots but not caviar?

Solution. Let $E$ be the event that a student enjoys escargots and $C$ be the event that a student enjoys caviar. Then $\Pr(E \cap C') = \Pr(E \cup C) - \Pr(C) = 0.85 - 0.54 = 0.31$

Alternatively, we can use the inclusion-exclusion principle to find

$$\begin{align*}
\Pr(E \cap C) &= \Pr(E) + \Pr(C) - \Pr(E \cup C) \\
\Pr(E \cap C) &= 0.75 + 0.54 - 0.85 \\
\Pr(E \cap C) &= 0.44
\end{align*}$$

And then

$$\Pr(E \cap C') = \Pr(E) - \Pr(E \cap C) = 0.75 - 0.44 = 0.31 \quad \text{//}$$
6 Graded Homework 4

1. Given that $E$ and $F$ are events with $\Pr(E) = 0.4$, $\Pr(F) = 0.45$, and $\Pr(E \cup F) = 0.75$, use a Venn diagram to find

(a) $\Pr(E \cap F) = 0.1$
(b) $\Pr(E' \cap F') = 0.25$

\[\begin{array}{c}
E \quad 0.3 \\
\text{0.75 - 0.45} \\
\text{0.75 - 0.4} \\
\text{0.25}
\end{array} \quad F \\
\text{1}\\n\]

2. An experiment has possible outcomes 1, 2, 3, 4, 5, and 6 which have the following probability distribution:

<table>
<thead>
<tr>
<th>outcome</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>0.28</td>
</tr>
<tr>
<td>5</td>
<td>0.07</td>
</tr>
<tr>
<td>6</td>
<td>0.14</td>
</tr>
</tbody>
</table>

(a) Find the probability that the outcome is 3.

Solution. Notice that

$$0.12 + 0.15 + 0.28 + 0.07 + 0.14 = 0.76$$

so the probability that the outcome is 3 is $1 - 0.76 = 0.24$. //

(b) If $E$ is the event “the outcome is even”, find $\Pr(E)$.

Solution. $\Pr(E) = 0.15 + 0.28 + 0.14 = 0.57$. //

3. For a new music video, a production company is to randomly order 4 cars from a pool of donated cars: 12 Lamborghini Huracáns and 7 Rolls-Royce Phantoms.

(a) What is the probability that no Phantoms are selected? Write your final answer as a reduced fraction.

Solution. The probability that no Phantoms are selected is $\frac{C(12, 4)}{C(19, 4)} = \frac{165}{1292}$. //
(b) What is the probability at least one Phantom is selected? Write your final answer as a reduced fraction.

Solution. The probability that at least one Phantom is selected is $1 - \frac{165}{1292} = \frac{1127}{1292}$. //

4. A single 6-sided die is rolled four times and the sequence of results is recorded.

(a) What is the probability of getting a sequence with four different numbers? Write your final answer as a reduced fraction.

Solution. $\frac{P(6, 4)}{6^4} = \frac{5}{18}$. //

(b) What is the probability of getting a sequence with exactly three 2’s? Write your final answer as a reduced fraction.

Solution. $\frac{C(4, 3) \cdot 5}{6^4} = \frac{5}{324}$. //

5. In Pleasant Stay Hotel, there are 15 floors. Suppose there are 5 guests on the elevator and that it is equally likely that a person will get off the elevator at any of the 15 floors. Find the probability that at least 2 passengers get off the elevator at the same floor. Write your final answer as a decimal rounded to 4 decimal places.

Solution. The probability that each of the passengers get off the elevator at different floors is $\frac{P(15, 5)}{15^5}$ so the probability that at least two passengers get off the elevator on the same floor is $1 - \frac{P(15, 5)}{15^5} \approx 0.5255$. //

6. The following table describes participants at Band Fest 2018.

<table>
<thead>
<tr>
<th>Music degree</th>
<th>Drummer</th>
<th>Bassist</th>
<th>Guitarist</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBA</td>
<td>33</td>
<td>15</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>19</td>
<td>7</td>
</tr>
</tbody>
</table>

Assume a participant is chosen at random.

(a) What is the probability the participant has an MBA, given that they are a Drummer? Write your final answer as a reduced fraction.

Solution. By the definition of conditional probability,

$$Pr(\text{MBA}|\text{Drummer}) = \frac{Pr(\text{MBA and Drummer})}{Pr(\text{Drummer})} = \frac{12}{33 + 12} = \frac{4}{15}$. //
(b) What is the probability the participant is a guitarist, given that they have a music degree? Write your final answer as a reduced fraction.

Solution. In a similar fashion,

\[ \Pr(\text{Guitarist}|\text{Music Deg.}) = \frac{27}{33 + 15 + 27} = \frac{9}{25}. \]

7. A bag contains 10 labeled balls: three are labeled with a 1, one is labeled with a 2, two are labeled with a 3, three are labeled with a 4, and one is labeled with a 5. A ball is selected at random. Let \( E \) be the event that “the number on the ball in even” and \( F \) be the event “the number on the ball is 4”.

Consider the visualization of the balls:

1 1 1 2 3 3 4 4 4 5

(a) Find \( \Pr(F|E) \). Write your final answer as a reduced fraction.

Solution. There are four balls labeled with an even number, three of which are labeled with a 4. So \( \Pr(F|E) = \frac{3}{4} \).

(b) Are the events \( E \) and \( F \) independent? Show your work to justify your answer.

Solution. First, notice that \( \Pr(F) = \frac{3}{10} \) and then, since

\[ \frac{3}{4} = \Pr(F|E) \neq \Pr(F) = \frac{3}{10}, \]

we see that \( E \) and \( F \) are not independent.

8. At SonicJuice, loudspeakers have three components: a woofer, a midrange driver, and a tweeter. The quality control department has determined that 2.3% of woofers, 3.5% of midrange drivers, and 1.7% of tweeters are defective. They have also determined that these defects are independent of each other.

(a) Find the probability that a randomly selected speaker has no defects. Write your answer in decimal form.

Solution. First, note that

- \( \Pr(\text{W good}) = 1 - 0.023 = 0.977 \)
- \( \Pr(\text{M good}) = 1 - 0.035 = 0.965 \)
- \( \Pr(\text{T good}) = 1 - 0.017 = 0.983 \)

By independence,

\[ \Pr(\text{W good} \cap \text{M good} \cap \text{T good}) = \Pr(\text{W good}) \cdot \Pr(\text{M good}) \cdot \Pr(\text{T good}) \]
\[ = (0.977)(0.965)(0.983) \]
\[ \approx 0.9268. \]

(b) Find the probability that a randomly selected speaker has exactly one defect. Write your answer in decimal form.
Solution. Exactly one defect occurs when either

- the woofer is defective but both the midrange driver and tweeter are fine
- the midrange driver is defective but both the woofer and tweeter are fine
- the tweeter is defective but both the woofer and midrange driver are fine

So the probability there is exactly one defect is

\[(0.023)(0.965)(0.983) + (0.035)(0.977)(0.983) + (0.017)(0.977)(0.965) \approx 0.07146. \]
7 Graded Homework 5

1. Recent research suggests that 73% of all shoulders have a chip on them. If 12 shoulders are selected at random and \( X \) is the random variable according to the number of shoulders with chips on them, fill in the blanks and use the information to find the probably that at most 4 shoulders have a chip on them. Write your final answer in decimal form rounded to four decimal places.

- Success = Shoulder with a chip on it
- \( n = 12 \)
- \( p = 0.73 \)
- \( q = 0.27 \)
- Possible values for the random variable \( X \): 0, 1, 2, \ldots, 12
- Probability that at most 4 shoulders have a chip on them:
  
  \[
  \text{binomcdf}(12, 0.73, 4) \approx 0.0047
  \]

2. The matrix \( A \) given below corresponds to a system of linear equations.

\[
A = \begin{bmatrix}
1 & 1 & -5 & 2 \\
0 & 2 & -6 & -2 \\
1 & -1 & 1 & 4
\end{bmatrix}
\]

After Gauss-Jordan elimination is applied to \( A \), the matrix \( B \) given below is obtained.

\[
B = \begin{bmatrix}
1 & 0 & -2 & 3 \\
0 & 1 & -3 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Circle the correct choice below and fill in any blank lines within your choice.

(B) There are infinitely many solutions where, for any real number \( z \), \( x = 2z + 3 \), \( y = 3z - 1 \)

3. Write the augmented matrix corresponding to the given system of linear equations. Solve the linear system using Gauss-Jordan elimination method by hand.

\[
\begin{align*}
4x + 2y & = 2 \\
-2x + 2y + 2z & = 4 \\
2x + y + z & = 8
\end{align*}
\]
Augmented matrix:
\[
\begin{bmatrix}
4 & 2 & 0 & 2 \\
-2 & 2 & 2 & 4 \\
2 & 1 & 1 & 8
\end{bmatrix}
\]

The Gauss-Jordan elimination is:
\[
\begin{bmatrix}
4 & 2 & 0 & 2 \\
-2 & 2 & 2 & 4 \\
2 & 1 & 1 & 8
\end{bmatrix}
\overset{\frac{1}{2} R_1}{\rightarrow}
\begin{bmatrix}
1 & 1/2 & 0 & 1/2 \\
-2 & 2 & 2 & 4 \\
2 & 1 & 1 & 8
\end{bmatrix}
\overset{R_2 + 2R_1}{\rightarrow}
\begin{bmatrix}
1 & 1/2 & 0 & 1/2 \\
0 & 3 & 2 & 5 \\
2 & 1 & 1 & 8
\end{bmatrix}
\overset{R_3 - 2R_1}{\rightarrow}
\begin{bmatrix}
1 & 1/2 & 0 & 1/2 \\
0 & 3 & 2 & 5 \\
0 & 0 & 1 & 7
\end{bmatrix}
\overset{\frac{1}{3} R_2}{\rightarrow}
\begin{bmatrix}
1 & 1/2 & 0 & 1/2 \\
0 & 1 & 2/3 & 5/3 \\
0 & 0 & 1 & 7
\end{bmatrix}
\overset{R_1 - \frac{1}{2} R_2}{\rightarrow}
\begin{bmatrix}
1 & 0 & -1/3 & -1/3 \\
0 & 1 & 2/3 & 5/3 \\
0 & 0 & 1 & 7
\end{bmatrix}
\overset{R_1 + \frac{1}{3} R_3}{\rightarrow}
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 2/3 & 5/3 \\
0 & 0 & 1 & 7
\end{bmatrix}
\overset{R_2 - \frac{2}{3} R_3}{\rightarrow}
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 7
\end{bmatrix}
\]

Solution:

\[x = 2, \ y = -3, \ z = 7\]

4. The CyberLotty is a game where 5 balls are randomly selected from a collection of 50 balls numbered 1 through 50. It costs 2 CryptoBlangs to play and the player guesses the 5 numbers to appear in the random selection. If the player guesses all 5 numbers correctly, they are awarded 1000 CryptoBlangs. If the player guesses 4 of the numbers correctly, they are awarded 100 CryptoBlangs. Otherwise, the player wins nothing. Find the expected net earning of this game and write your solution in decimal form rounded to three decimal places.

Solution. First, notice that the selection process splits the 50 numbers into 5 winning numbers and 45 losing numbers. There are

- \(C(50, 5) = 2118760\) ways to select 5 numbers from 50,
- \(C(5, 5) = 1\) ways to select 5 of the 5 winning numbers,
- \(C(5, 4) \cdot C(45, 1) = 225\) ways to match 4 of the winning numbers (and 1 of the losing numbers),
- \(2118760 - (225 + 1) = 2118534\) ways to match 3 numbers or less.
Then the awards paired with their correspond probability is

<table>
<thead>
<tr>
<th>prize</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>( \frac{1}{2118760} )</td>
</tr>
<tr>
<td>100</td>
<td>( \frac{225}{2118760} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{2118534}{2118760} )</td>
</tr>
</tbody>
</table>

The net earnings paired with their corresponding probability is

<table>
<thead>
<tr>
<th>net earnings</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>998</td>
<td>( \frac{1}{2118760} )</td>
</tr>
<tr>
<td>98</td>
<td>( \frac{225}{2118760} )</td>
</tr>
<tr>
<td>−2</td>
<td>( \frac{2118534}{2118760} )</td>
</tr>
</tbody>
</table>

Then the expected value of the net earnings can be computed as follows:

\[
\left( 1000 \cdot \frac{1}{2118760} + 100 \cdot \frac{225}{2118760} + 0 \cdot \frac{2118534}{2118760} \right) - 2 \approx -1.989
\]

or

\[
998 \cdot \frac{1}{2118760} + 98 \cdot \frac{225}{2118760} + (-2) \cdot \frac{2118534}{2118760} \approx -1.989
\]
8 Graded Homework 6

1. Perform the matrix multiplication. Show your work for full credit.

\[
\begin{bmatrix}
2 & 1 & 4 \\
-1 & 0 & 1/2 \\
0 & 6 & 1
\end{bmatrix}
\begin{bmatrix}
6 & -1 & 0 \\
6 & 3 & 3 \\
-6 & 0 & 2
\end{bmatrix}
= 
\begin{bmatrix}
-6 & 1 & 11 \\
-9 & 1 & 1 \\
30 & 18 & 20
\end{bmatrix}
\]

which results from

\[
\begin{array}{ccc}
2 \cdot 6 + 1 \cdot 6 + 4 \cdot (-6) & 2 \cdot (-1) + 1 \cdot 3 + 4 \cdot 0 & 2 \cdot 0 + 1 \cdot 3 + 4 \cdot 2 \\
(-1) \cdot 6 + 0 \cdot 6 + (1/2)(-6) & (-1)(-1) + 0 \cdot 3 + (1/2) \cdot 0 & (-1) \cdot 0 + 0 \cdot 3 + (1/2) \cdot 2 \\
0 \cdot 6 + 6 \cdot 6 + 1 \cdot (-6) & 0 \cdot (-1) + 6 \cdot 3 + 1 \cdot 0 & 0 \cdot 0 + 6 \cdot 3 + 1 \cdot 2 \\
\end{array}
\]

2. Determine whether or not the matrices

\[
\begin{bmatrix}
8 & 3 & -4 \\
-6 & -2 & 3 \\
-3 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
-1 & -1 & -1 \\
3 & 4 & 0 \\
0 & 1 & -2
\end{bmatrix}
\]

are inverses of each other. Show work to receive credit.

**Solution.** Since

\[
\begin{bmatrix}
8 & 3 & -4 \\
-6 & -2 & 3 \\
-3 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -1 & -1 \\
3 & 4 & 0 \\
0 & 1 & -2
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
6 & 8 & 1
\end{bmatrix}
\neq 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

these two matrices are not inverses of each other. //

3. Research shows that if a person buys a Pear laptop there is a 92% probability that their next laptop purchase will also be from Pear and if a person buys a Mirrors laptop there is a 57% probability that their next laptop purchase will be from Mirrors. Indicate the transition diagram and the transition matrix. Use appropriate labels.

**Solution.** Notice that there is an 8% probability that a person with a Pear laptop will switch to Mirrors and a 43% probability that a person with a Mirrors laptop will switch to a Pear.

The corresponding transition diagram is

The corresponding transition matrix is

\[
\begin{bmatrix}
0.92 & 0.43 \\
0.08 & 0.57
\end{bmatrix}
\]

\[
\begin{array}{c|cc}
\text{current} & P & M \\
\hline
\text{next} & P & 0.92 & 0.43 \\
& M & 0.08 & 0.57
\end{array}
\]

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4. Use the Gauss-Jordan method (showing each step of your work) to find the inverse of

\[
\begin{bmatrix}
1 & -1 & 1 \\
-1 & 2 & -2 \\
-1 & 2 & -1
\end{bmatrix}
\]

**Solution.**

\[
\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & 0 \\
-1 & 2 & -2 & 0 & 1 & 0 \\
-1 & 2 & -1 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{R_2 + R_1}
\begin{bmatrix}
1 & -1 & 1 & 1 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
-1 & 2 & -1 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{R_3 + R_1}
\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\xrightarrow{R_1 + R_2}
\begin{bmatrix}
1 & 0 & 0 & 2 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\xrightarrow{R_3 - R_2}
\begin{bmatrix}
1 & 0 & 0 & 2 & 1 & 0 \\
0 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1
\end{bmatrix}
\xrightarrow{R_2 + R_3}
\begin{bmatrix}
1 & 0 & 0 & 2 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 1
\end{bmatrix}
\]

So the inverse matrix is

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 0 & 1 \\
0 & -1 & 1
\end{bmatrix}
\]

5. Write the system of equations

\[
\begin{cases}
x - y + z = 2 \\
-x + 2y - 2z = 3 \\
-x + 2y - z = 7
\end{cases}
\]

as a matrix equation and use the inverse matrix of the coefficient matrix to solve the system.

**Solution.** Let

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
-1 & 2 & -2 \\
-1 & 2 & -1
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

19
and

\[
C = \begin{bmatrix} 
2 \\
3 \\
7 
\end{bmatrix}
\]

which gives the matrix equation

\[AX = C.\]

In problem 4, we found

\[
A^{-1} = \begin{bmatrix} 
2 & 1 & 0 \\
1 & 0 & 1 \\
0 & -1 & 1 
\end{bmatrix}
\]

so

\[
A^{-1}AX = A^{-1}C
\]

\[
X = A^{-1}C
\]

and

\[
A^{-1}C = \begin{bmatrix} 
7 \\
9 \\
4 
\end{bmatrix}
\]

That is, \(x = 7\), \(y = 9\), and \(z = 4\). //
9 Test 2 Extra Credit Questions

1. Two tests are taken to determine readiness for Psychokinesis training. Research shows that any person has a 43% chance of passing Test I, a 22% chance of passing Test II, and a 55.54% chance of passing at least one of the tests. Is passing Test I independent of passing Test II? Justify your answer.

*Solution.* By the Inclusion-Exclusion Principle,

\[
\Pr(I \cup II) = \Pr(I) + \Pr(II) - \Pr(I \cap II)
\]

which gives

\[
0.5554 = 0.43 + 0.22 - \Pr(I \cap II)
\]

Since

\[
\Pr(I) \cdot \Pr(II) = 0.0946 = \Pr(I \cap II),
\]

passing Test I is independent of passing Test II. ///

2. Two tests are taken to determine readiness for Precognition training. Research shows that any person has a 45% chance of passing Test I, a 19% chance of passing Test II, and a 59.25% chance of passing at least one of the tests. Is passing Test I independent of passing Test II? Justify your answer.

*Solution.* By the Inclusion-Exclusion Principle,

\[
\Pr(I \cup II) = \Pr(I) + \Pr(II) - \Pr(I \cap II)
\]

which gives

\[
0.5925 = 0.45 + 0.19 - \Pr(I \cap II)
\]

Since

\[
\Pr(I) \cdot \Pr(II) = 0.0855 \neq \Pr(I \cap II),
\]

passing Test I is not independent of passing Test II. ///

3. A student walks into their Astrology class to find that they are taking an exam. The exam consists of 20 questions. Each question is multiple choice with 4 possible answer choices, only one choice of which is correct. If they randomly guess on all of the problems, what is the probability they make a 70% or better?

*Solution.* To get a 70% or better, this student needs to get at least 14 questions correct. Since the student is randomly guessing, this is a binomial experiment where \(n = 20\) and \(p = 1/4\) where the *success* condition is guessing a question correctly. The complementary event is getting at most 13 correct which can be computed using the `binomcdf` function in the calculator. That is, if \(X\) is the number of questions the
students guesses correctly, the probability that this student guesses 14 questions or more correctly is

\[
\Pr(X \geq 14) = 1 - \Pr(X < 14) \\
= 1 - \Pr(X \leq 13) \\
= 1 - \text{binomcdf}(20, 1/4, 13) \\
\approx 0.0000295117
\]

4. A new book randomizing machine has

- 10 copies of Dante’s Inferno,
- 5 copies of Foucault’s Discipline and Punish, and
- 2 copies of Vonnegut’s Slaughterhouse-Five.

Professor Fletcher is going to use the book randomizing machine to assign one book to each student in a class of 15 students.

(a) What is the probability that no student is assigned Vonnegut’s book?

**Solution.** First, we note the possible ways in which to assign the books in terms of how many of each book we use:

<table>
<thead>
<tr>
<th>Dante</th>
<th>Foucault</th>
<th>Vonnegut</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Now, we count the number of ways we can make each kind of assignment:

- \(10D, 5F, 0V\) \(\rightarrow C(15, 10) \cdot C(5, 5) \cdot C(0, 0) = 3003\)
- \(10D, 4F, 1V\) \(\rightarrow C(15, 10) \cdot C(5, 4) \cdot C(1, 1) = 15015\)
- \(10D, 3F, 2V\) \(\rightarrow C(15, 10) \cdot C(5, 3) \cdot C(2, 2) = 30030\)
- \(9D, 5F, 1V\) \(\rightarrow C(15, 9) \cdot C(6, 5) \cdot C(1, 1) = 30030\)
- \(9D, 4F, 2V\) \(\rightarrow C(15, 9) \cdot C(6, 4) \cdot C(2, 2) = 75075\)
- \(8D, 5F, 2V\) \(\rightarrow C(15, 8) \cdot C(7, 5) \cdot C(2, 2) = 135135\)

\[
\text{sum} = 288288
\]

The only way in which no student is assigned Vonnegut’s book is in the \(10D, 5F, 0V\) arrangement. Thus, the probability that no student is assigned Vonnegut’s book is

\[
\frac{3003}{288288} = \frac{1}{96} \approx 0.01042
\]
(b) Given that exactly one copy of Vonnegut’s book has been assigned, what is the probability that all five copies of Foucault’s book have been assigned?

Solution. With our above considerations, we know that exactly one of Vonnegut’s books is assigned in either the 10D,4F,1V or the 9D,5F,1V arrangement. So the total number of ways in which exactly one of Vonnegut’s books is assigned is

\[ 15015 + 30030 = 45045. \]

Of these, 30030 are the ways in which exactly 5 of Foucault’s book get assigned. Therefore,

\[ \Pr(5F\mid 1V) = \frac{\Pr(5F \text{ and } 1V)}{\Pr(1V)} = \frac{30030}{45045} = \frac{2}{3} \approx 0.666667. \]

5. The MegaLot is a game where 5 balls are randomly selected from a bag of white balls numbered 1 through 55. Then one ball is randomly selected from a bag of red balls numbered 1 through 100. Each participant guesses the 5 numbers to appear on the white balls and the 1 number to appear on the red ball and prizes are awarded as follows:

- If all 6 numbers are correct, the lucky winner is awarded $5,000,000.
- If the 5 numbers selected from the white balls are correct but the red ball is incorrect, the winner is awarded $10,000.
- Otherwise, the player wins nothing.

It costs $3 to play the MegaLot.

(a) Find the expected net earnings.

Solution. First, we count the total number of possible winning selections:

\[ C(55, 5) \cdot C(100, 1) = 347,876,100 \]

Once the selection is made, the balls are then split into the following piles:

<table>
<thead>
<tr>
<th></th>
<th>winning</th>
<th>losing</th>
</tr>
</thead>
<tbody>
<tr>
<td>white</td>
<td>5</td>
<td>50</td>
</tr>
<tr>
<td>red</td>
<td>1</td>
<td>99</td>
</tr>
</tbody>
</table>

- There is \[ C(5, 5) \cdot C(1, 1) = 1 \] way to get all 6 numbers correct.
- There are \[ C(5, 5) \cdot C(99, 1) = 99 \] ways to get all 5 white correct and the red one wrong.
There are \(347,876,100 - (1 + 99) = 347,876,000\) ways to not be a prize winner.

Therefore, the expected net earnings are

\[
\left(5,000,000 \cdot \frac{1}{347,876,100} + 10,000 \cdot \frac{99}{347,876,100} + 0 \cdot \frac{347,876,000}{347,876,100}\right) - 3 \approx -2.983
\]

(b) Given that you have a winning ticket, what is the probability you got all 6 numbers correct?

\textit{Solution.} There are \(1 + 99 = 100\) ways to get a winning ticket as we calculated above. Of these, only 1 of them corresponds to getting all numbers correct. Therefore, the probability that your ticket has all 6 numbers correct given that it is winning is

\[
\frac{1}{100}
\]