What is Mathematics?

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(Q1) What is a mathematical object?
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A list of examples should include:

- the natural numbers
- the real numbers
- groups
- functions
(Q1) What is a mathematical object?

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- the real numbers
- groups
- functions

The search for foundations has found enthusiastic productivity in Set Theory. In particular, one can give a formal ground to (most? all?) of Mathematics with the “primitive” notions of *classes* and *sets*. 
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All mathematical objects are those things which can be described in terms of Set Theory.

From an ontological point of view, what we’re saying here is that anything that is a mathematical object can be “reduced to” or “completely characterized” by sets and classes.
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An alternative representation:

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\begin{array}{c}
0 \leftrightarrow \\
1 \leftrightarrow /\\
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\vdots & \\
\end{align*}
\]

Are these qualitatively different representations? What is the “true nature” of the natural numbers (if such a question is even coherent)?
Now, if the numbers aren’t controversial enough, let’s turn our attention to *ordered pairs*. Anyone trained after the adoption of R. Descartes’ Cartesian plane has an informal notion (probably somehow geometric) of the ordered pair \( \langle x, y \rangle \).
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The above notion was introduced by K. Kuratowski around 1921 but other attempts were made, including the suggestion around 1914 by F. Hausdorff that the pair \( \langle x, y \rangle \) be identified with \( \{\{x, 1\}, \{y, 2\}\} \).
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N. Wiener proposed his own representation in 1914 following the monumental *Principia Mathematica* written by B. Russell and A.N. Whitehead published in volumes across 1910 to 1913.
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One may suggest that this “more natural”-ness is really a matter of aesthetics and/or psychology.
To recapitulate, the program previously outlined is that of formalizing our informal notions (particularly in the context of Mathematics).
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So what can we say about the informal notions?
Let’s take a step back:
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What does it mean to exist?
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Maybe an answer can be captured by answering

What exists?
Some potential examples of things that exist:

- “mathematical” objects (the natural numbers, the real numbers, groups, etc.)
- “physical (visible)” objects (a chair, human bodies, etc.)
- “physical (invisible)” objects (electrons, quarks, strings, etc.)
- “emergent” objects (sand dunes, physical center of mass, etc.)
- “conceptual” objects (personal/subjective identity, feelings/emotions, colors, Truth, Justice, etc.)

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Perhaps these tentative “categories” of potential existents aren’t actually distinct or perhaps some exist and others don’t.
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The claim is here is twofold:

- Mathematical objects exist.
- Anything that exists is a mathematical object.
Some motivation to accept such a claim:
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Cosmologists (and even laypeople) can ‘model’ or conceive of the universe in terms of some (4,5,11?)-dimensional manifold. So, if this universe of “stuff” (for the moment adopting a monist attitude) and a lot of (most? all?) its perceivable phenomena can be explained in such terminology (and we have no other equally useful competing model), why should we look any further?
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If all things in the human experience can be described by a finite collection of properties (and these beings are finite beings), then we can even “reduce” them to sets or other mathematical objects like dynamical systems. (For instance, an apple may be fully characterized by its spacio-temporal relation to other objects, color, shape, taste, etc.)
This table is a set.
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CC (me!) is a dynamical system: my body is a space (perhaps a manifold or even a discrete space) where it (and all of its constituents like organs and hormones) undergoes continuous transformations in time (continuous or discrete).
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Everything is math!
But what about potentially imperceivable existents?
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What about things outside the scope of Math and Physics?
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What about things outside the scope of Math and Physics?

- Psychology?
- Justice?
- Personhood?
If everything were Mathematics, then E. Wigner’s discussion in his “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [2] is completely incoherent.
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Of course Mathematics is useful in studying Mathematics!
About a decade later, E. Wigner expresses sentiments in “Physics and the Explanation of Life” [3] that, in some uncertain future, fields like Biology and Psychology may find levels of rigor comparable to that of Physics.
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What we must do is develop an understanding of *animate* objects with such depth as we have learned to understand *inanimate* objects and it seems fields like Mathematical Biology are giving a breath of life to E. Wigner’s suspicion.
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What about the relation “is modeled ‘effectively’ by”?  

In blissful youth (or in a Pre-Calculus class), one sees that the trajectory of a thrown rock through space can be effectively modeled by a continuous function (maybe even a parabola in the most pristine of cases!).
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After the rock completes its short journey, our model of its trajectory also has *descriptive* power. That is, we can explain previously observed phenomena with our model.
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But is the relation “is modeled effectively by” an equivalence relation?
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(E’1) Everything that exists (and is perceivable by humans) is modeled effectively by Mathematics (mathematical objects).

Now E. Wigner’s Unreasonableness makes sense since, with E’1, we are not necessarily asserting that mathematical objects exist (are “out there” in the world).
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But what is a mathematical description?
P. Jackson, in “Introduction to Artificial Intelligence” [1], offers a suggestion:

- A **finite** set of undefined *words* or *symbols* (objects, relations between objects, logical connectives, etc.)
- A **finite** set of *sentences* based on these words and symbols (axioms, postulates)
- A **finite** set of *logical rules* (still based on the words and symbols) which tell us how to produce other sentences from the axioms
For example, let’s consider the Peano Axioms for the Natural Numbers.
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We start with our finite set of words and symbols:

number, 0, =, x, y, z, S, is, for every, implies, iff, and...
Now, our axioms:

- 0 is a number
- for ever number $x$, $x = x$
- for all numbers $x$ and $y$, $x = y$ implies $y = x$
- for all numbers $x$, $y$, and $z$, $x = y$ and $y = z$ implies $x = z$
- for every number $x$ and for any $y$, $x$ is a number and $x = y$ implies $y$ is a number
- for every number $x$, $S(x)$ is a number
- for all numbers $x$ and $y$, $x = y$ iff $S(x) = S(y)$
- for every number $x$, it is not the case that $S(x) = 0$
As far as the logical rules go, we rely on *modus ponens*:

Given sentences $\phi$ and $\psi$: if $\phi$ is a *true* true sentence and “$\phi$ implies $\psi$” is a *true* sentence, $\psi$ is a *true* sentence.
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Suppose we have some infinite graph (immersed in the plane) which cannot be constructed from a finite collection of rules. That is, the edge relation is sufficiently complicated to not be understood from any finite collection of rules for construction.

How can one, as a human being, begin to describe or understand such a graph?
With P. Jackson’s formulation of what constitutes a mathematical description, we immediately see that any mathematical description is a finite description. In other words, something which is mathematically described is finitely describable.
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As made clear from Peano’s Axioms, just because something is finitely describable doesn’t mean that it must be finite.
With P. Jackson’s formulation of what constitutes a mathematical description, we immediately see that any mathematical description is a finite description. In other words, something which is mathematically described is \textit{finitely describable}.

As made clear from Peano’s Axioms, just because something is finitely describable doesn’t mean that it must be finite.

Moreover, the theory of the thing being described (the collection of deducible sentences) need not be finite either. What the mathematical description offers is a coherent way to formally talk about some aspects of the objects. Such a description may not be “encompassing” in the sense that any question that can be asked about the object necessarily have an answer according to the theory.
Now let’s turn our attention to things which are *finitely describable*. 
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The assertion of the previous question can be seen as an informal expression of the Church-Turing Thesis.
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- From this theory, we can only derive \textit{countably} many sentences.
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We suggest that there are things which are not finitely describable:

- Suppose we have a finite theory which offers a finite description of real numbers.
- From this theory, we can only derive countably many sentences.
- Since the collection of real numbers is not countable, there must be some real number which is not describable in this theory.
While we can’t offer a finite theory which yields a finite description of every real number, there are ways to finitely describe the collection of real numbers as a whole (resorting to second-order statements).
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If we attempt to offer finite descriptions of things like Beauty, Justice, Subjective Identity, etc., a Philosophy course discussing the relevant notion will reveal the labyrinth of seemingly insurmountable difficulties in such a project.
So what are these “things” that populate our minds and senses and how can we establish a coherent discourse in which to situate them?
A list of suggested authors:

- Luitzen Egbertus Jan (L.E.J.) Brouwer
- Penelope Maddy
- Hilary Putnam
- Gian-Carlo Rota
- David Ruelle
- Robert S. Tragesser
- Stewart Shapiro
- Eugene Wigner

Only to name a few...
