NOTES ON UNIVERSALLY NULL SETS

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Here, we summarize some results regarding universally null subsets of Polish spaces and conclude with the fact that, following J. Mycielski, one can produce an analytic subfield of $\mathbb{R}$ which is not universally null. Many results come from [7] and [8].

1. INTRODUCTION AND PRELIMINARIES

For a Polish space $X$, we will let $\mathcal{B}(X)$ denote the class of Borel subsets of $X$. If $X$ is Polish, then the space of Borel probability measures $\mathcal{M}(X)$ on $X$ is Polishable with the weak topology. A continuous measure $\mu \in \mathcal{M}(X)$ is a measure giving every finite set measure zero. Let $\mathcal{M}_c(X)$ be the collection of continuous $\mu \in \mathcal{M}(X)$. Then $\mathcal{M}_c(X)$ is a dense $G_\delta$ subset of $\mathcal{M}(X)$ as long as $X$ has no isolated points. Refer to [10] for more details.

**Definition 1.** Let $X$ be a Polish space. We say that $A \subseteq X$ is universally null if $\mu^*(A) = 0$ for every $\mu \in \mathcal{M}_c(X)$ where $\mu^*$ is the outer measure generated by $\mu$.

**Definition 2.** Let $X$ be a Polish space. For $A \subseteq X$, we define the annihilator of $A$ to be
\[
\mathcal{N}(A) = \{ \mu \in \mathcal{M}(X) : \mu^*(A) = 0 \}.
\]
We say that $A$ is residually null if $\mathcal{N}(A)$ is co-meager in $\mathcal{M}(X)$.

For a more in depth discussion of the theory of residually null sets, refer to [3]. Of immediate interest is that all universally null subsets of any Polish space without isolated points are also residually null.

**Definition 3.** Let $X$ be a topological space. We say that a set $A \subseteq X$ is a Luzin set if it is uncountable and, for every $M \subseteq X$ which is meager, $A \cap M$ is countable.

**Theorem 4** (Mahlo [6], Luzin [5]). Assuming CH, there is a set $A \subseteq \mathbb{R}$ which is a Luzin set.

**Proof.** Let $\{F_\alpha : \alpha < \omega_1\}$ be the collection of all closed nowhere dense subsets of $\mathbb{R}$. For each $\alpha < \omega_1$, inductively choose
\[
x_\alpha \not\in \{x_\beta : \beta < \alpha\} \cup \bigcup \{ F_\beta : \beta \leq \alpha \}.
\]
Then the set $\{x_\alpha : \alpha < \omega_1\}$ is as desired. \qed

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One can immediately see that the set produced in Theorem 4 is not meager.

**Definition 5 (Borel [2]).** Let $X$ be a metric space with compatible metric $ho$. We say that $A \subseteq X$ has **strong measure zero** if, given any sequence \( \{\varepsilon_n > 0 : n \in \omega\} \), there exists a sequence \( \{A_n \subseteq X : n \in \omega\} \) so that \( \text{diam}_{\rho}(A_n) < \varepsilon_n \) and \( A \subseteq \bigcup\{A_n : n \in \omega\} \).

In a metric space \((X, \rho)\), since \( \text{diam}_{\rho}(A) = \text{diam}_{\rho}(\text{cl}(A)) \), we can cover any strong measure zero set with a countable union of closed sets satisfying the requisite properties.

**Definition 6 (Besicovitch [1]).** Let $X$ be a topological space. We say that $A \subseteq X$ is **concentrated on** $B \subseteq X$ provided that, for every open set $U$ so that $B \subseteq U$, $A \setminus U$ is countable.

**Theorem 7 (Szpiłrajn [14]).** Let $X$ be a Polish space. A set $A \subseteq X$ is a Luzin set if and only if $A$ is uncountable and concentrated on every countable dense set of $X$.

**Proof.** ($\Rightarrow$) Suppose $A$ is a Luzin set and $D \subseteq X$ is countable and dense. Let $U$ be any open set with $D \subseteq U$. Then $U$ is dense so $X \setminus U$ is closed and nowhere dense. It follows that $A \cap (X \setminus U) = A \setminus U$ is countable as $A$ is a Luzin set.

($\Leftarrow$) Now, suppose $A$ is uncountable and concentrated on every countable dense subset $Q$ of $X$. We need only check that $A$ meets every meager set on a countable set. It suffices to check that $A$ meets every closed nowhere dense set on a countable set so let $F \subseteq X$ be closed and nowhere dense. Let $U = X \setminus F$ and notice that $D := Q \cap U$ is a countable dense set. Since $U$ is open and $D \subseteq U$, we have that $A \setminus U$ is countable. That is, $A \setminus (X \setminus F) = A \cap F$ is countable, finishing the proof. \(\square\)

**Theorem 8 (Sierpiński [11]).** Let $X$ be a Polish space and suppose $A \subseteq X$ is concentrated on a countable dense set. Then $A$ has strong measure zero.

**Proof.** Let $D = \{d_n : n \in \omega\}$ be dense in $X$ so that $A$ is concentrated on $D$. Now, let $\{\varepsilon_n > 0 : n \in \omega\}$ and, for $n \in \omega$, pick an open set $U_{2n}$ that contains $d_n$ and satisfies $\text{diam}(U_{2n}) < \varepsilon_{2n}$. Define $U = \bigcup\{U_{2n} : n \in \omega\}$ and, since $A$ is concentrated on $D$ and $D \subseteq U$, notice that $A \setminus U$ is countable. Enumerate $A \setminus U = \{x_n : n \in \omega\}$ and, for $n \in \omega$, let $U_{2n+1}$ be an open set containing $x_n$ which satisfies $\text{diam}(U_{2n+1}) < \varepsilon_{2n+1}$. Finally, notice that the family $\{U_n : n \in \omega\}$ covers $A$ and has the property that, for each $n \in \omega$, $\text{diam}(U_n) < \varepsilon_n$. Therefore, $A$ has strong measure zero. \(\square\)

**Lemma 9.** Let $X$ be a metric space and $\mu$ be a Borel probability measure on $X$. If $\mu$ is continuous, then, for every $\varepsilon > 0$ and $x \in X$, there exists a neighborhood $U_x$ of $x$ with $\mu(U_x) < \varepsilon$.
Lemma 10. Let \((X, \rho)\) be a compact metric space and \(\mu\) be a continuous Borel probability measure on \(X\). Then, for any \(\varepsilon > 0\), there exists a \(\delta > 0\) so that, for every closed \(F \subseteq X\) with \(\text{diam}_\rho(F) < \delta\), \(\mu(F) < \varepsilon\).

**Proof.** For each \(x \in X\), appeal to Lemma 9 to pick \(r_x > 0\) so that \(B_\rho(x, 2r_x)\) satisfies \(\mu(B_\rho(x, 2r_x)) < \varepsilon\). As \(X\) is compact, pick a finite subset \(A \subseteq X\) so that \(\{B_\rho(x, r_x) : x \in A\}\) covers \(X\). Define \(\delta = \min\{r_x : x \in A\} > 0\).

Now, let \(F \subseteq X\) be any closed set so that \(\text{diam}_\rho(F) < \delta\). Fix \(y \in F\) and pick \(x \in A\) so that \(y \in B_\rho(x, r_x)\). Now, for any \(z \in F\), notice that

\[
\rho(x, z) \leq \rho(x, y) + \rho(y, z) < r_x + \delta \leq 2r_x.
\]

That is, \(z \in B_\rho(x, 2r_x)\) and, as \(z \in F\) was arbitrary, we see that \(F \subseteq B_\rho(x, 2r_x)\) which provides us with the fact that \(\mu(F) < \varepsilon\). \(\square\)

**Theorem 11** (Szpilrajn [13]). Let \(X\) be a Polish space and \(A \subseteq X\) have strong measure zero. Then \(A\) is universally null.

**Proof.** Suppose \(A \subseteq X\) is strong measure zero and let \(\mu\) be any continuous Borel probability on \(X\). Let \(\varepsilon > 0\) be arbitrary and define \(\varepsilon_n = \frac{\varepsilon}{2^{n+2}}\) for \(n \in \omega\). By [10, Theorem II.3.2] we can find compact \(K \subseteq X\) so that

\[
\mu(K) > \max\left\{1 - \frac{\varepsilon}{2^n}, 0\right\}.
\]

Now, notice that \(\nu\) defined on the Borel sets of \(K\) by \(\nu(E) = \frac{\mu(E)}{\mu(K)}\) is a Borel probability measure on \(K\). So, by Lemma 10, we can pick \(\delta_n > 0\) so that, whenever \(F \subseteq K\) is closed with \(\text{diam}(F) < \delta_n\), we have that \(\nu(F) < \varepsilon_n\).

Consider \(A' = A \cap K\) and notice that \(A'\) is also strong measure zero. Thus, we can find \(\{A_n \subseteq K : n \in \omega\}\) consisting of closed sets so that \(\text{diam}(A_n) < \delta_n\) and \(A' \subseteq \bigcup\{A_n : n \in \omega\}\). It follows that

\[
\nu^*(A') \leq \sum_{n \in \omega} \nu(A_n) \leq \sum_{n \in \omega} \frac{\varepsilon}{2^{n+2}} = \frac{\varepsilon}{2}.
\]

Hence,

\[
\frac{\mu^*(A')}{\mu(K)} = \nu^*(A') \leq \frac{\varepsilon}{2} \implies \mu^*(A') \leq \frac{\varepsilon}{2} \cdot \mu(K) \leq \frac{\varepsilon}{2}.
\]

Now, note that

\[
1 - \frac{\varepsilon}{2} < \mu(K) \implies \mu(X \setminus K) < \frac{\varepsilon}{2}.
\]

Finally, we have that \(\mu^*(A) \leq \mu^*(A \setminus K) + \mu^*(A') < \varepsilon\). As \(\varepsilon\) was arbitrary, we see that \(\mu^*(A) = 0\), completing the proof. \(\square\)

**Corollary 12.** Every Luzin subset of a Polish space is universally null.

**Proof.** Combine Theorems 7, 8, and 11. \(\square\)

**Theorem 13.** Assuming CH, there exists a set \(A \subseteq \mathbb{R}\) which is universally null which fails to have the Baire property.
Proof. Use Theorem 4 to produce a Luzin set $A \subseteq \mathbb{R}$ and notice that Corollary 12 guarantees that $A$ is universally null. Moreover, $A$ is residually null and, since $A$ is non-meager, we appeal to [3, Theorem 14] to conclude that $A$ cannot have the Baire property. \hfill \square

Lemma 14 (Sierpiński-Szpilrajn [12]). Let $X$ be a Polish space, $A \subseteq X$, and $\mathcal{B}_A = \{ B \cap A : B \in \mathcal{B}(X) \}$. Then the following are equivalent.

(i) $A$ is universally null;
(ii) any continuous measure on $(A, \mathcal{B}_A)$ is identically zero.

Proof. ((i) $\Rightarrow$ (ii)) Suppose $\mu$ is a continuous measure on $(A, \mathcal{B}_A)$. Define $\nu$ on $(X, \mathcal{B}(X))$ by the rule

$$\nu(B) = \mu(B \cap A)$$

and observe that $\nu$ is a measure on $X$. Moreover, as $\mu$ is continuous, $\nu$ is continuous. Hence, $\nu^*(A) = 0$. From this we see that $\mu$ must be identically zero.

((ii) $\Rightarrow$ (i)) Let $\mu$ be a continuous Borel probability measure on $X$. Now, define $\nu$ on $(A, \mathcal{B}_A)$ by the rule

$$\nu(E) = \inf \{ \mu(B) : B \in \mathcal{B}(X), E \subseteq B \}.$$ 

To see that $\nu$ is a measure on $(A, \mathcal{B}_A)$, let $\{E_n : n \in \omega\} \subseteq \mathcal{B}_A$ be a pair-wise disjoint family. For each $n \in \omega$, pick $B_n^* \in \mathcal{B}(X)$ so that $E_n = A \cap B_n^*$ and $E_n^* \in \mathcal{B}(X)$ so that $E_n \subseteq E_n^*$ and $\mu(E_n) = \nu(E_n)$. Then let $B_n = B_n^* \cap E_n^*$.

Now,

$$E := \bigcup \{E_n : n \in \omega\} \subseteq \bigcup \{B_n : n \in \omega\} =: B$$

and $B \in \mathcal{B}(X)$. For $n, m \in \omega$ with $n \neq m$, notice that

$$B_n \cap B_m \cap A = (B_n \cap A) \cap (B_m \cap A)$$

$$\subseteq (B_n^* \cap A) \cap (B_m^* \cap A)$$

$$= E_n \cap E_m$$

$$= \emptyset.$$ 

From this, we see that $E_n \subseteq B_n \setminus B_m$ which gives

$$\mu(B_n) = \nu(E_n) \leq \mu(B_n \setminus B_m) \leq \mu(B_n) = \mu(B_n \setminus B_m) + \mu(B_n \cap B_m).$$

Particularly, $\mu(B_n \cap B_m) = 0$ for $n \neq m$. It follows that

$$\nu(E) \leq \mu(B) = \sum_{n \in \omega} \mu(B_n) = \sum_{n \in \omega} \nu(E_n).$$
For the other inequality, pick \( G \in \mathcal{B}(X) \) so that \( G \cap A = E \) and \( \mu(G) = \nu(E) \). Then, since \( E_k \subseteq E \cap B_k \subseteq G \cap B_k \),

\[
\sum_{k=0}^{n-1} \nu(E_k) \leq \sum_{k=0}^{n-1} \mu(G \cap B_k) = \mu((G \cap B_0) \cup \cdots \cup (G \cap B_{n-1})) \\
\leq \mu(G) = \nu(E).
\]

Since this inequality holds for arbitrary \( n \in \omega \),

\[
\sum_{n \in \omega} \nu(E_n) \leq \nu(E)
\]

which establishes that \( \nu \) is a measure on \((A, \mathcal{B}_A)\).

As \( \nu \) is a continuous measure on \((A, \mathcal{B}_A)\), by hypothesis, \( \nu \) is identically zero. This guarantees that \( \mu^*(A) = 0 \), finishing the proof. \( \square \)

**Lemma 15.** Let \( X \) be Polish and \( A \subseteq X \). If \( A \) is universally null, then any continuous measure on \( \wp(A) \) is identically zero.

**Proof.** Let \( \mu \) be a continuous measure on \((A, \wp(A))\). Since \( \mu \rceil \wp_A \) is a measure on \((A, \mathcal{B}_A)\), \( \mu \rceil \wp_A \) is identically zero. Now, for any \( E \subseteq A \) and \( B \in \mathcal{B}_A \) with \( E \subseteq B \), since \( \mu(E) \leq \mu(B) = 0 \), we see that \( \mu \) is identically zero. \( \square \)

**Proposition 16.** The class of universally null subsets of a Polish space \( X \) is invariant under Borel isomorphisms.

**Proof.** Immediate from Lemma 14. \( \square \)

**Theorem 17** (Szpilrajn). The product of universally null sets is universally null.

**Proof.** Let \( X \) and \( Y \) be universally null and suppose \( \mu \) is a continuous Borel probability measure on \( X \times Y \). Define a measure \( \nu \) on \( Y \) by

\[
\nu(B) = \mu(X \times B)
\]

for each Borel set \( B \subseteq Y \). Now, since \( X \) is universally null, \( \nu(\{y\}) = \mu(X \times \{y\}) = 0 \) for every \( y \in Y \) so \( \nu \) is continuous. It follows that, as \( Y \) is also universally null, \( \nu(Y) = \mu(X \times Y) = 0 \), finishing the proof. \( \square \)

**Lemma 18.** Suppose \( G \subseteq \mathbb{R} \) is co-meager. Then, for any \( z \in \mathbb{R} \), there exist \( x, y \in G \) so that \( z = x + y \).

**Proof.** Let \( z \in \mathbb{R} \) and, since \( G \) is co-meager, so is \( z - G \). Now, let \( y \in G \cap (z - G) \) and pick \( x \in G \) so that \( y = z - x \). \( \square \)

**Theorem 19** (Sierpiński). Assuming CH, there exists a universally null set which is not strong measure zero.
Proof. Let \( \{ F_\alpha : \alpha < \omega_1 \} \) be the collection of all closed nowhere dense subsets of \( \mathbb{R} \) and \( \{ z_\alpha : \alpha < \omega_1 \} \) be an enumeration of \( \mathbb{R} \). Pick \( x_0, y_0 \notin F_0 \) so that \( x_0 + y_0 = z_0 \). For each \( 0 < \alpha < \omega_1 \), inductively choose \( x_\alpha, y_\alpha \notin \{ x_\beta : \beta < \alpha \} \cup \{ y_\beta : \beta < \alpha \} \cup \bigcup \{ F_\beta : \beta \leq \alpha \} \) with the property that \( x_\alpha + y_\alpha = z_\alpha \). Then define \( X = \{ x_\alpha : \alpha < \omega_1 \} \cup \{ y_\alpha : \alpha < \omega_1 \} \).

Now we argue that \( X^2 \) is universally null but not strong measure zero. As \( X \) is transparently a Luzin set, it is universally null by Corollary 12. Hence, by Theorem 17, \( X^2 \) is universally null in \( \mathbb{R}^2 \). Now, define \( p : \mathbb{R}^2 \to \mathbb{R} \) by the rule \( p(x, y) = x + y \). Fix \( a, b \in \mathbb{R} \) and let \( x, y \in \mathbb{R} \) be so that \( \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon \) where \( \varepsilon > 0 \). Now,

\[
\begin{align*}
| (x + y) - (a + b) |^2 &= (x - a + y - b)^2 \\
&= (x - a)^2 + 2(x - a)(y - b) + (y - b)^2 \\
&< 3 \cdot \varepsilon^2.
\end{align*}
\]

From this, we see that \( p \) takes \( \varepsilon \)-balls to \( \sqrt{3} \cdot \varepsilon \)-balls. Hence, \( p \) preserves strong measure zero sets. As \( p[X]^2 = \mathbb{R} \), it can’t possibly be the case that \( X^2 \) is strong measure zero.

\[\square\]

**Theorem 20** (Grzegorek, Ryll-Nardzewski [4]). There exists a universally measurable set which is not a Borel set modulo a universally null set.

**Proof.** The example here is WO. Since \( \text{WO} \in \Pi^1_1 \), it is universally measurable. Now let \( B \) be a Borel set. We will show that

\[ \text{WO} \Delta B \]

is not universally null. Notice that \( B \setminus \text{WO} \in \Sigma^1_1 \). So if \( B \setminus \text{WO} \) were uncountable, it would contain a perfect set ensuring that \( \text{WO} \Delta B \) is not universally null. So \( B \setminus \text{WO} \) must be countable.

Now, without loss of generality, we can assume that \( B \subseteq \text{WO} \). Since \( \text{WO} \) is \( \Pi^1_1 \)-complete, \( \text{WO} \setminus B \) is \( \Pi^1_1 \)-complete. Moreover, \( \text{WO} \setminus B \) is uncountable. Hence, \( \text{WO} \setminus B \) has the perfect set property so \( \text{WO} \setminus B \) is not universally null.

\[\square\]

2. **Subfields of \( \mathbb{R} \)**

**Theorem 21** (Mycielski [9]). For any meager set \( Y \) of irrationals, there exists an uncountable perfect set \( X \) so that the field generated by \( X \) is disjoint from \( Y \).

**Corollary 22.** There exists an analytic subfield \( F \) of \( \mathbb{R} \) which is not universally null. Incidentally, such an \( F \) is meager.

**Proof.** Let \( Y \) be a meager set of irrationals, and \( X \) be an uncountable perfect set so that the field \( F \) generated by \( X \) is disjoint from \( Y \) by Theorem 21.
Immediately, as $F$ is the field generated by $X$ and $X$ is closed, $F$ is analytic. The fact that $F$ supports a non-degenerate Borel probability measure follows from the fact that $F$ contains an uncountable perfect set. That is, $F$ is not universally null.

Now, since $F$ is analytic, it has the Baire property. As $F$ contains the rationals, $F$ is dense. If $F$ were non-meager, it would be open. But then, as an open subgroup of $\mathbb{R}$, $F$ would also be closed yielding $F = \mathbb{R}$, a contradiction. Therefore, $F$ is meager. □

REFERENCES