Transfinite Dimension Theory

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For a regular space $X$, ind$X$ is defined recursively as follows:

- $\text{ind} X = -1$, $X = \emptyset$;
- $\text{ind} X \leq n$, $(\forall x \in X)(\forall U \in \mathcal{N}_x)(\exists V \in \mathcal{N}_x)$
  
  $[V \subseteq U \land \text{ind} \partial V < n]$;
- $\text{ind} X = n$, $\text{ind} X \leq n \land (\forall m < n)[\text{ind} X \leq m \text{ fails}]$.

**Fact.** $\text{ind} \mathbb{R}^n = n$ for $n \in \mathbb{N}$. 

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Definition

For a separable metric space $X$, we define $dX$ recursively as follows:

- $dX = -1$, $X = \emptyset$;
- $dX = 0$, $X$ has a basis of clopen sets;
- $dX \leq \alpha$, $X = Z \cup \bigcup \{F_n : n < \omega\}$ where $dZ \leq 0$, each $F_n$ closed and $dF_n < \alpha$;
- $dX = \alpha$, $dX \leq \alpha$ and $\forall \beta < \alpha$, $dX \leq \beta$ fails.
Results

Theorem
\(d\) is a topological invariant.

Theorem
For \(n < \omega\), \(dX = n\) if and only if \(\text{ind}X = n\).

Theorem
If \(dX\) exists, \(dX\) is a countable ordinal.
Theorem (Subspace Theorem)
If $Y \subseteq X$ and $dX$ exists, then $dY$ exists and $dY \leq dX$.

Theorem
If $dX = \alpha$, then, for any $\beta \leq \alpha$, there is a closed subset $F \subseteq X$ so that $dF = \beta$.

Theorem (Countable Sum Theorem)
If $X = \bigcup\{F_n : n < \omega\}$ where each $F_n$ is closed, then $dX = \sup\{dF_n : n < \omega\}$. 
For an ordinal $\alpha$, we let
\[ \lambda(\alpha) = \sup\{\beta : \beta \leq \alpha \text{ and } \beta \text{ is a limit ordinal}\} \] and $n(\alpha) < \omega$ the unique natural so that $\alpha = \lambda(\alpha) + n(\alpha)$. Now, define $\Sigma : \text{ON}^2 \rightarrow \text{ON}$ by the rule

\[ \Sigma(\alpha, \beta) = \max\{\lambda(\alpha), \lambda(\beta)\} + \min\{\lambda(\alpha), \lambda(\beta)\} + n(\alpha) + n(\beta). \]

**Theorem (Product Theorem)**

Suppose $X$ and $Y$ are so that $dX$ and $dY$ exist. Then $d(X \times Y) \leq \Sigma(dX, dY)$.

**Theorem (Addition Theorem)**

Suppose $X$ and $Y$ are so that $dX$ and $dY$ exist. Then $d(X \cup Y) \leq \Sigma(dX, dY) + 1$. 
For a dimension function $e$ and a limit ordinal $\alpha$, we say that $X$ is **strongly $\alpha$-dimensional** if $X = \bigcup \{ F_n : n < \omega \}$ where each $F_n$ is closed and $eF_n < \alpha$.

**Theorem**

Let $\lambda$ be a limit ordinal, $n \in \mathbb{N}$ and suppose $dX$ exists. Then $dX \leq \lambda + n$ if and only if $X = X_\lambda \cup X_n$ where $X_\lambda$ is strongly $\lambda$-dimensional and $dX_n \leq n$. 
If $e$ is a finite dimension function on the class of separable metrizable spaces that satisfies the following, then $e = \text{ind}$.

N.I. $e\{\emptyset\} = 0$;
N.II. $Y \subseteq X$ implies that $eY \leq eX$;
N.III. if $X = \bigcup\{F_n : n < \omega\}$ where each $F_n$ is closed, $eX = \sup\{eF_n : n < \omega\}$;
N.IV. $e(X \cup Y) \leq eX + eY + 1$;
N.V. if $eX < \omega$, then there is a compactification $\hat{X}$ of $X$ so that $e\hat{X} = eX$;
N.VI. if $X \neq \emptyset$ and $eX < \omega$, then there is a base $\mathcal{B}$ for $X$ so that, for every $U \in \mathcal{B}$, $e\partial U \leq eX - 1$. 
I. \( e\{\emptyset\} = 0; \)

II. \( Y \subseteq X \) implies that \( eY \leq eX; \)

III. if \( X = \bigcup\{F_n : n < \omega\} \) where each \( F_n \) is closed, 
    \( eX = \sup\{eF_n : n < \omega\}; \)

IV. \( e(X \cup Y) \leq \Sigma(eX, eY) + 1; \)

V. if \( eX \) exists, then there is a compactification \( \hat{X} \) of \( X \) so that \( e\hat{X} = eX; \)

VI. if \( X \neq \emptyset \) and \( eX \) exists, then there is a base \( B \) for \( X \) so that, for every \( U \in B, \)
    \( e\partial U \leq eX - 1 \) when \( eX \) is a successor or
    \( \partial U \) is strongly \( eX \)-dimensional when \( eX \) is a limit.
Questions

In [2], a metric continuum is constructed which has $d$ dimension $\omega + 1$.

*Question.* Does there exist a separable metric space $X$ with $dX \geq \omega + 2$?

*Question.* Does (V) hold? Do (I - VI) characterize the $d$ dimension?
References

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