TOPOLOGICAL ENTROPY AND MEAN TOPOLOGICAL DIMENSION

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1. Topological and Dynamical Preliminaries

Notation. In general, the convention $\mathbb{N} = \{0, 1, 2, \ldots \}$ is held and we will let $\mathbb{N}_1 = \{1, 2, 3, \ldots \}$.

Lemma 1. If a sequence is sub-additive, i.e., $a_{n+m} \leq a_n + a_m$, then

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf \frac{a_n}{n}.$$

Proof. Let $m$ and $n$ be arbitrary. Then there are unique numbers $q$ and $r < m$ so that $n = mq + r$. Observe that

$$\frac{a_n}{n} = \frac{a_{mq+r}}{n} \leq \frac{amq}{mq+r} + \frac{ar}{n} \leq \frac{am}{m} + \max\{a_i : i < m\}.$$

Hence,

$$\limsup \frac{a_n}{n} \leq \frac{am}{m}.$$

As $m$ was arbitrary, we see that

$$\limsup \frac{a_n}{n} \leq \inf \frac{a_n}{n},$$

which finishes the proof. \qed

Definition 2. Let $X$ be a topological space. For open covers $\mathcal{U}$ and $\mathcal{V}$, we see that $\mathcal{V}$ refines $\mathcal{U}$ if, $(\forall V \in \mathcal{V})(\exists U \in \mathcal{U})(V \subseteq U)$.

Definition 3. Let $X$ be a topological space and let $\mathcal{U}$ be an open cover of $X$. We define the order of $\mathcal{U}$ to be

$$\text{ord}(\mathcal{U}) = -1 + \max \left\{ \sum_{U \in \mathcal{U}} 1_U(x) : x \in X \right\}.$$

Then we define $\mathcal{D}(\mathcal{U}) = \min\{\text{ord}(\mathcal{V}) : \mathcal{V} \text{ refines } \mathcal{U}\}$.

Remark. Notice that $\text{ord}(\mathcal{U}) \leq -1 + \# \mathcal{U}$.

Definition 4. The topological dimension of a topological space $X$ is defined to be

$$\dim X = \sup\{\mathcal{D}(\mathcal{U}) : \mathcal{U} \text{ is a finite open cover of } X\}.$$

Remark. For this definition, it takes some work, but it can be shown that $\dim \mathbb{R}^n = n$.

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Definition 5. Let $X$ be a topological space. For open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$, we define
$$
\bigvee_{i=1}^n \mathcal{U}_i = \{U_1 \cap U_2 \cap \cdots \cap U_n : U_j \in \mathcal{U}_j\}.
$$

Definition 6. Let $X$ be a topological space and $T : X \to X$ be continuous. For a family $\mathcal{A}$ consisting of subsets of $X$, we let $T^{-1}[\mathcal{A}] = \{T^{-1}[A] : A \in \mathcal{A}\}$.

For an open cover $\mathcal{U}$ of $X$ and $n \in \mathbb{N}$, we define
$$
\mathcal{U}^T_n = \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U}].
$$

Notation. For topological spaces $X$ and $Y$, let $C(X,Y)$ denote the collection of continuous functions from $X$ to $Y$.

Definition 7. By a dynamical system we mean a pair $(X,T)$ where $X$ is a compact metrizable space and $T \in C(X,X)$ is continuous.

Definition 8. Let $T \in C(X,X)$. We say that a point $x \in X$ is a $T$-wandering point if there exists a neighborhood $U$ of $x$ and $n \in \mathbb{N}$ so that, for every $m \geq n$, $T^m[U] \cap U = \emptyset$. Let
$$
\Omega(X,T) = \{x \in X : x \text{ is not a } T\text{-wandering point}\}.
$$

When a dynamical system $(X,T)$ is understood, we may let $\Omega = \Omega(X,T)$.

Proposition 9. Let $(X,T)$ be a dynamical system and let $\Omega$ be the set of non-wandering points. Then $\Omega$ is closed and $T$-invariant so $(\Omega,T|_\Omega)$ is a dynamical system.

Proof. It is clear that the set of $T$-wandering points is open so $\Omega$ is closed. The only thing left to show is that $T[\Omega] \subseteq \Omega$. To this end, suppose $x \in X$ is so that $T(x) \not\in \Omega$. Let $U$ and $n$ witness that $T(x)$ is wandering and consider the neighborhood $T^{-1}[U]$ of $x$. Let $k \geq n$ and notice that $T^k[T^{-1}[U]] = T^{k-1}[U]$. Now, observe that
$$
T[T^{-1}[U] \cap T^{k-1}[U]] \subseteq U \cap T^k[U] = \emptyset,
$$
as $k \geq n$. It follows that $T^{-1}[U] \cap T^{k-1}[U] = \emptyset$ for all $k \geq n$. Hence, $x \not\in \Omega$. Therefore, if $x \in \Omega$, $T(x) \in \Omega$ which is exactly that $T[\Omega] \subseteq \Omega$.

Proposition 10. Let $(X,T)$ be a dynamical system with an implicit compatible metric. Let $W$ be the set of $T$-wandering points and suppose $K \subseteq W$ is compact. Then there exists $\beta > 0$ so that, for all $x \in K$ and all $k \in \mathbb{N}_{\geq 1}$, $T^k[B(x,\beta)] \cap B(x,\beta) = \emptyset$. 
Proof. Let \( d \) be the implicit compatible metric on the space. Let \( y \in K \) and let \( U_y \) be a neighborhood of \( y \) witness that \( y \) is wandering and let

\[
    n_y = \min \{ m : \forall k \geq m \ T^k[U_y] \cap U_y = \emptyset \}.
\]

Let \( r_y > 0 \) be so that \( B(y, r_y) \subseteq U_y \). Notice that \( T^i(y) \neq y \) for each \( 0 < i < n_y \). Let \( \varepsilon_y = \min \{ d(y, T^i(y)) : 0 < i < n_y \} \). As \( T^2 \) is continuous, let \( \gamma_i > 0 \) be so that, when \( d(y, z) < \gamma_i, \ d(T^i(y), T^i(z)) < \frac{\varepsilon_y}{2} \). Now, let \( \gamma_y = \min \{ \gamma_i : 0 < i < n_y \} \). Define

\[
    \beta_y = \min \left\{ \frac{r_y, \gamma_y, \frac{\varepsilon_y}{2}}{2} \right\}.
\]

Let \( A \subseteq K \) be finite so that \( \{ B(y, \beta_y) : y \in A \} \) covers \( K \) and let

\[
    \beta = \min \{ \beta_y : y \in A \}.
\]

Now, we just need to check that \( \beta \) is as desired. Let \( x \in K \) be arbitrary and \( k \in \mathbb{N} \). Pick \( y \in A \) so that \( x \in B(y, \beta_y) \). Let \( z \in B(x, \beta) \) be arbitrary and consider the following cases.

Case 1: \( k \geq n_y \). Notice that

\[
    d(y, z) \leq d(y, x) + d(x, z) < \beta_y + \beta \leq 2\beta_y \leq r_y.
\]

Hence, \( z \in U_y \). It follows that \( T^k(z) \notin U_y \). As \( z \in B(x, \beta) \) was arbitrary, \( T^k[B(x, \beta)] \cap B(x, \beta) = \emptyset \).

Case 2: \( 0 < k < n_y \). Observe that

\[
    d(y, z) \leq d(y, x) + d(x, z) < 2\beta_y \leq \gamma_y.
\]

It follows that \( d(T^k(y), T^k(z)) < \frac{\varepsilon_y}{2} \). Observe that

\[
    \varepsilon_y \leq d(y, T^k(y)) \leq d(y, x) + d(x, T^k(z)) + d(T^k(z), T^k(y)) \leq \beta_y + d(x, T^k(z)) + \frac{\varepsilon_y}{2}.
\]

As \( \beta_y \leq \frac{\varepsilon_y}{2} \), we have that

\[
    \beta \leq \beta_y \leq \frac{\varepsilon_y}{4} < d(x, T^k(z)).
\]

That is, \( T^k(z) \notin B(x, \beta) \). Again, as \( z \in B(x, \beta) \) was arbitrary, we see that \( T^k[B(x, \beta)] \cap B(x, \beta) = \emptyset \). \( \square \)

2. Topological Entropy

The following notion of topological entropy first appeared in [1].

**Definition 11.** Let \( X \) be a compact metrizable space. For an open cover \( \mathcal{U} \) of \( X \), let

\[
    \mathcal{N}(\mathcal{U}) = \min \{ \# \mathcal{V} : \mathcal{V} \text{ is a finite sub-cover of } \mathcal{U} \}.
\]

Then, the **entropy** of \( \mathcal{U} \) is \( H(\mathcal{U}) := \log \mathcal{N}(\mathcal{U}) \). For \( T \in \mathcal{C}(X, X) \) and an open cover \( \mathcal{U} \), we define

\[
    h(T, \mathcal{U}) = \lim_{n \to \infty} \frac{H(\mathcal{U}^n)}{n}
\]
and
\[ h(T) = \sup \{ h(T, \mathcal{U}) : \mathcal{U} \text{ is a finite open cover of } X \}. \]

We call \( h(T) \) the entropy of \( T \).

**Proposition 12.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be finite open covers \( X \). Let \( T \in C(X,X) \). Then \( N(\mathcal{U} \cup \mathcal{V}) \leq N(\mathcal{U}) N(\mathcal{V}) \) and \( N(T^{-n}[\mathcal{U}]) \leq N(\mathcal{U}) \).

**Proof.** Let \( \mathcal{U}_0 \) and \( \mathcal{V}_0 \) be finite sub-covers of \( \mathcal{U} \) and \( \mathcal{V} \), respectively, so that \( \# \mathcal{U}_0 = N(\mathcal{U}) \) and \( \# \mathcal{V}_0 = N(\mathcal{V}) \). Notice that \( \mathcal{U}_0 \cup \mathcal{V}_0 \) is a finite sub-cover of \( \mathcal{U} \cup \mathcal{V} \). Also, notice that \( \#(\mathcal{U}_0 \cup \mathcal{V}_0) \leq \# \mathcal{U}_0 \cdot \# \mathcal{V}_0 \).

So \( N(\mathcal{U} \cup \mathcal{V}) \leq N(\mathcal{U}) N(\mathcal{V}) \).

Let \( \mathcal{W} \) be a finite sub-cover of \( \mathcal{U} \) so that \( \# \mathcal{W} = N(\mathcal{U}) \). Notice that \( \# T^{-n}[\mathcal{W}] \leq \# \mathcal{W} \) and that \( T^{-n}[\mathcal{W}] \) is a finite sub-cover of \( T^{-1}[\mathcal{U}] \). Hence, \( N(T^{-n}[\mathcal{U}]) \leq N(\mathcal{U}) \).

The previous proposition guarantees that \( N \) is sub-multiplicative which gives us that \( H \) is sub-additive. Thus, entropy is defined.

**Example 13.** Let \( K \) be a compact metrizable space and let \( \sigma : K^\mathbb{N} \leftarrow \mathbb{N} \) be the standard shift map. If \( K = n \), then \( h(\sigma) = \log n \). Otherwise (\( K \) is an infinite set) we have that \( h(\sigma) = \infty \).

**Proposition 14.** Let \((X,T)\) be a dynamical system. For \( k \geq 1 \), \( h(T^k) = k \cdot h(T) \).

The following notions of entropy are due to Rufus Bowen.

**Definition 15.** Let \((X,T)\) be a dynamical system and \( d \) a compatible metric on \( X \). For \( n \in \mathbb{N}_1 \), define
\[ d^T_n(x,y) = \max \{ d(T^m x, T^m y) : m < n \}. \]

**Proposition 16.** For any \( n \in \mathbb{N}_1 \), \( d^T_n \) is a compatible metric for \( X \).

**Definition 17.** Let \((X,T)\) be a dynamical system with compatible metric \( d \) and \( S \subseteq X \). For \( n \in \mathbb{N}_1 \) and \( \varepsilon > 0 \), we say that \( S \) is \((n,\varepsilon,d,T)\)-separated provided that
\[ (\forall (x,y) \in S^2 \setminus \Delta)(d^T_n(x,y) \geq \varepsilon) \]
where \( \Delta = \{(x,x) : x \in X\} \). Let \( A \subseteq X \). We define
\[ R_{\text{sep}}(A,n,\varepsilon,d,T) = \sup \{ \#S : S \subseteq A \text{ is } (n,\varepsilon,d,T)\text{-separated} \}. \]

Since we will often be making choices for such a set, let \( S_{\text{sep}}(A,n,\varepsilon,d,T) \subseteq A \) be so that
\[ \#S_{\text{sep}}(A,n,\varepsilon,d,T) = R_{\text{sep}}(A,n,\varepsilon,d,T). \]

Then we define
\[ H_{\text{sep}}(A,\varepsilon,d,T) = \limsup_{n \to \infty} \frac{\log R_{\text{sep}}(A,n,\varepsilon,d,T)}{n}. \]
When the metric $d$ and map $T$ are understood in context, we will often suppress their use in the parameters. For example, we will write $R_{\text{sep}}(A, n, \varepsilon)$ in place of $R_{\text{sep}}(A, n, \varepsilon, d, T)$.

**Proposition 18.** Let $X$ be metrizable, $Y$ be metrizable and $f : X \to Y$ be continuous. Then $f$ is uniformly continuous.

*Proof.* Let $\varepsilon > 0$ be arbitrary. Let $x \in X$, and pick $\delta(x) > 0$ so that

$$d(x, y) < 2\delta(x) \implies d(f(x), f(y)) < \frac{\varepsilon}{2}.$$  

Pick $x_0, \ldots, x_n$ be so that $\{B(x_i, \delta(x_i)) : i \leq n\}$ is a cover of $X$. Let $\delta = \min\{\delta(x_i) : i \leq n\}$. Now, suppose $d(x, y) < \delta$. Let $x_i$ be so that $x \in B(x_i, \delta(x_i))$. Observe that

$$d(x_i, y) \leq d(x_i, x) + d(x, y) < \delta(x_i) + \delta \leq 2\delta(x_i).$$

It follows that

$$d(f(y), f(x_i)) < \frac{\varepsilon}{2} \text{ and } d(f(x), f(x_i)) < \frac{\varepsilon}{2}.$$  

Therefore, $d(f(x), f(y)) < \varepsilon$, finishing the proof. \qed

**Proposition 19.** $R_{\text{sep}}(A, n, \varepsilon, d, T) < \infty$.

*Proof.* First, observe that $T^i$ is uniformly continuous for each $0 < i < n$. So we can pick $\delta_i > 0$ to be so that, for $x$ and $y$ with $d(x, y) < 2\delta_i$, $d(T^i(x), T^i(y)) < \varepsilon$. Let $\delta = \min\{\delta_i : 0 < i < n\}$. Let $\mathcal{V}$ be a finite subcover of $\{B(x, \delta) : x \in X\}$ and suppose $S \subseteq X$ is any $(n, \varepsilon, d, T)$-separated set. Define $\phi : S \to \mathcal{V}$ by letting $\phi(x) \in \mathcal{V}$ be so that $x \in \phi(x)$. We can define $\phi$ since $\mathcal{V}$ is an open cover. To see that $\phi$ is an injection, notice that if $x, y \in V \in \mathcal{V}$, we have that $d(x, y) < 2\delta$ which implies that $d(T^i(x), T^i(y)) < \varepsilon$ for all $i < n$. As $S$ is $(n, \varepsilon, d, T)$-separated, we have that $x = y$. Hence, $\#S \leq \#\mathcal{V}$. As $S$ was arbitrary, we see that $R_{\text{sep}}(A, n, \varepsilon, d, T) \leq \#\mathcal{V}$, finishing the proof. \qed

**Definition 20.** Let $(X, T)$ be a dynamical system, $d$ a compatible metric, and $A \subseteq X$. For $n \in \mathbb{N}_1$ and $\varepsilon > 0$, we will say that $S \subseteq A$ is an $(n, \varepsilon, d, T)$-spanning for $A$, or that $S (n, \varepsilon, d, T)$-spans $A$, provided that

$$(\forall x \in A)(\exists y \in S)(d^n(x, y) < \varepsilon).$$

Then we define

$$R_{\text{span}}(A, n, \varepsilon, d, T) = \min\{\#S : S (n, \varepsilon, d, T)\text{-span } A\}.$$  

Let $S_{\text{span}}(A, n, \varepsilon, d, T) \subseteq A$ so that

$$\#S_{\text{span}}(A, n, \varepsilon, d, T) = R_{\text{span}}(A, n, \varepsilon, d, T).$$

Then we define

$$H_{\text{span}}(A, \varepsilon, d, T) = \limsup_{n \to \infty} \frac{\log R_{\text{span}}(A, n, \varepsilon, d, T)}{n}.$$
Like before, when the metric $d$ and map $T$ are understood, we will suppress their use in the parameters.

**Remark.** Notice that, by our definition, a set $S$ is $(n, \varepsilon, d, T)$-spanning for $A$ if and only if

$$\{B_{d_n^T}(x, \varepsilon) : x \in S\}$$

is a cover of $A$.

**Proposition 21.**

$$R_{\text{sep}}(A, n, 2\varepsilon, d, T) \leq R_{\text{span}}(A, n, \varepsilon, d, T) \leq R_{\text{sep}}(A, n, \varepsilon, d, T).$$

**Proof.** Let $S \subseteq A$ be $(n, \varepsilon, d, T)$-separated and suppose $x \in A$ is so that, for all $y \in S$, $d_n^T(x, y) \geq \varepsilon$. It follows that $S \cup \{x\}$ is $(n, \varepsilon, d, T)$-separated. Hence, $S_{\text{sep}}(A, n, \varepsilon, d, T)$ is an $(n, \varepsilon, d, T)$-spanning for $A$. Thus,

$$R_{\text{span}}(A, n, \varepsilon, d, T) \leq \#S_{\text{sep}}(A, n, \varepsilon, d, T) = R_{\text{sep}}(A, n, \varepsilon, d, T).$$

Now, consider the sets $S_{\text{sep}}(A, n, 2\varepsilon, d, T)$ and $S_{\text{span}}(A, n, \varepsilon, d, T)$. Define

$$\phi : S_{\text{sep}}(A, n, 2\varepsilon, d, T) \rightarrow S_{\text{span}}(A, n, \varepsilon, d, T)$$

by choosing $\phi(x)$ be so that $d_n^T(x, \phi(x)) < \varepsilon$. Suppose $\phi(x) = \phi(y)$ for $x, y \in S_{\text{sep}}(A, n, 2\varepsilon, d, T)$.

Then, for $i < n$,

$$d(T^i x, T^i y) \leq d(T^i x, T^i \phi(x)) + d(T^i \phi(x), T^i y)$$

$$\leq d_n^T(x, \phi(x)) + d_n^T(\phi(x), y)$$

$$< 2\varepsilon.$$
Definition 23. Let $X$ be a metrizable space. For a compatible metric $d$ and a real number $\epsilon > 0$ we define

$$N(\epsilon, d) = \min\{\#U : \text{$U$ is an open cover of $X$ with mesh}_d U < \epsilon\}.$$ 

Next, for $T \in C(X, X)$, we define

$$R_T(\epsilon, d) = \limsup_{n \to \infty} \frac{\log N(\epsilon, d^T_n)}{n}.$$ 

Proposition 24. Let $\epsilon \leq \delta$ and $d$ be any compatible metric on $X$. Then $N(\delta, d) \leq N(\epsilon, d)$. Moreover, $R_T(\delta, d) \leq R_T(\epsilon, d)$.

Proof. Let $d$ be any compatible metric on $X$ and let $\mathcal{U}$ be an open cover of $X$ with mesh$_d \mathcal{U} < \epsilon$ and $\#\mathcal{U} = N(\epsilon, d)$. Notice that mesh$_d \mathcal{U} < \delta$. Hence, $N(\delta, d) \leq \#\mathcal{U} = N(\epsilon, d)$. Now, let $n \in \mathbb{N}_1$ and notice that, since $d^T_n$ is a compatible metric on $X$,

$$\frac{\log N(\delta, d^T_n)}{n} \leq \frac{\log N(\epsilon, d^T_n)}{n}.$$ 

Taking $n \to \infty$, we see that $R_T(\delta, d) \leq R_T(\epsilon, d)$. \hfill $\square$

Remark. From this proposition we see that $R_T(\epsilon, d)$ is increasing as $\epsilon \to 0$. Hence,

$$\sup\{R_T(\epsilon, d) : \epsilon \geq 0\} = \lim_{\epsilon \to 0} R_T(\epsilon, d).$$

Proposition 25. $R_{\text{span}}(X, n, 3\epsilon, d, T) \leq N(3\epsilon, d^T_n) \leq R_{\text{span}}(X, n, \epsilon, d, T)$.

Proof. Let $S = S_{\text{span}}(X, n, \epsilon, d, T)$. Define $\mathcal{U} = \{B_{d^T_n}(x, \epsilon) : x \in S\}$. Notice that $\mathcal{U}$ is an open cover of $X$ as $S$ is $(n, \epsilon, d, T)$-spanning and that $\#\mathcal{U} = \#S$. Notice that, for any $x \in S$, diam$_{d^T_n} B_{d^T_n}(x, \epsilon) \leq 2\epsilon$. It follows that mesh$_{d^T_n} \mathcal{U} \leq 2\epsilon < 3\epsilon$. Hence,

$$N(3\epsilon, d^T_n) \leq \#S = R_{\text{span}}(X, n, \epsilon, d, T).$$

Let $\mathcal{U}$ be an open cover so that mesh$_{d^T_n} \mathcal{U} < 3\epsilon$ and $\#\mathcal{U} = N(3\epsilon, d^T_n)$. For each $U \in \mathcal{U}$, let $x_U \in U$. We will now show that $\{x_U : U \in \mathcal{U}\}$ is $(n, 3\epsilon, d, T)$-spanning. Let $y \in X$ and let $U \in \mathcal{U}$ be so that $y \in U$. It follows that

$$d^T_n(y, x_U) \leq \text{diam}_{d^T_n}(U) \leq \text{mesh}_{d^T_n} \mathcal{U} < 3\epsilon.$$ 

Thus,

$$R_{\text{span}}(X, n, 3\epsilon, d, T) \leq \#\{x_U : U \in \mathcal{U}\} \leq \#\mathcal{U} = N(3\epsilon, d^T_n),$$

finishing the proof. \hfill $\square$

Before we can show that all of these notions provide us with information about topological entropy, we must introduce the Lebesgue number.

Definition 26. Let $X$ be a metrizable space and $d$ a compatible metric. For an open cover $\mathcal{U}$ of $X$, define the **Lebesgue number** of $\mathcal{U}$ with respect to the metric $d$ to be

$$\mathcal{L}_d(\mathcal{U}) = \sup\{r \geq 0 : (\forall A \subseteq X) (\text{diam}_d(A) < r \Rightarrow (\exists U \in \mathcal{U})(A \subseteq U))\}.$$
Proposition 27. Let \( \mathcal{U} \) be an open cover of \( X \) and \( \mathcal{V} \) be any open cover of \( X \) with the property that \( \text{mesh}_d \mathcal{V} < L_d(\mathcal{U}) \). Then \( \mathcal{V} \) refines \( \mathcal{U} \).

Lemma 28. Let \( X \) be a compact metrizable space with compatible metric \( d \). Let \( \mathcal{U} \) be any open cover of \( X \). Then \( L_d(\mathcal{U}) > 0 \).

Proof. Let \( x \in X \) and pick \( U_x \in \mathcal{U} \) so that \( x \in U_x \). Let \( \varepsilon(x, U_x) > 0 \) be so that \( B_d(x, 2\varepsilon(x, U_x)) \subseteq U_x \). Since \( \{ B_d(x, \varepsilon(x, U_x)) : x \in X \} \) is an open cover of \( X \) and \( X \) is compact, let \( \{ x_i : i < n \} \) be a collection of points so that \( \{ B_d(x_i, \varepsilon(x_i, U_{x_i})) : i < n \} \) is a cover of \( X \). For convenience, let \( \varepsilon_i = \varepsilon(x_i, U_{x_i}) \). Now, let \( \varepsilon = \min \{ \varepsilon_i : i < n \} \) and let \( A \subseteq X \) be so that \( \text{diam}_d(A) < \varepsilon \). Fix \( x \in A \) and pick \( i < n \) so that \( x \in B_d(x_i, \varepsilon_i) \). For any \( y \in A \), observe that

\[
d(x_i, y) \leq d(x_i, x) + d(x, y) < \varepsilon_i + \varepsilon \leq 2\varepsilon_i.
\]

It follows that \( A \subseteq B_d(x_i, 2\varepsilon_i) \subseteq U_{x_i} \). Therefore, \( L_d(\mathcal{U}) \geq \varepsilon > 0 \). \( \square \)

Proposition 29. For any compatible metric \( d \), we have that

\[
h(T) = \lim_{\varepsilon \to 0} R_T(\varepsilon, d) = \lim_{\varepsilon \to 0} H_{\text{sep}}(X, \varepsilon, d, T) = \lim_{\varepsilon \to 0} H_{\text{span}}(X, \varepsilon, d, T).
\]

Proof. Combining Propositions 21 and 25, we see that

\[
\lim_{\varepsilon \to 0} R_T(\varepsilon, d) = \lim_{\varepsilon \to 0} H_{\text{sep}}(X, \varepsilon, d, T) = \lim_{\varepsilon \to 0} H_{\text{span}}(X, \varepsilon, d, T).
\]

Let \( \varepsilon > 0 \) and let \( \mathcal{U} \) be a finite sub-cover of

\[
\{ B_d(x, \varepsilon/3) : x \in X \}.
\]

Let \( A \in \mathcal{U}^n \). That is, there are \( U_0, U_1, \ldots, U_{n-1} \in \mathcal{U} \) so that

\[
A = U_0 \cap T^{-1}[U_1] \cap \cdots \cap T^{-n+1}[U_{n-1}].
\]

Now, to see that the \( d_n^{T_d} \)-diameter of \( A \) is less than \( \varepsilon \), let \( x, y \in A \). For \( j \leq n - 1 \), \( T^j x, T^j y \in U_j \) which gives us that \( d(T^j x, T^j y) \leq \frac{\varepsilon}{2^j} \varepsilon < \varepsilon \). It follows that \( d_n^{T_d}(x, y) < \varepsilon \). As \( A \) was arbitrary, we see that \( \text{mesh}_{d_T} \mathcal{U}^n < \varepsilon \).

Now, let \( \mathcal{V} \) be a finite sub-cover of \( \mathcal{U}^n \) so that \#(\mathcal{V}) = \#(\mathcal{U}^n). \) We also have that \( \mathcal{V} \) is a cover of \( X \) with \( \text{mesh}_{d_T} (\mathcal{V}) < \varepsilon \) so

\[
N(\varepsilon, d_n^{T_d}) \leq \#(\mathcal{V}) = N(\mathcal{U}^n).
\]

It follows that

\[
R_T(\varepsilon, d) \leq h(T, \mathcal{U}) \leq h(T).
\]

Hence,

\[
\lim_{\varepsilon \to 0} R_T(\varepsilon, d) = \sup\{ R_T(\varepsilon, d) : \varepsilon > 0 \} \leq h(T).
\]

Let \( \mathcal{U} \) be a finite open cover and let \( \varepsilon = L_d(\mathcal{U})/3 \). For notational convenience, let \( S_n = S_{\text{sep}}(X, n, \varepsilon) \). Let \( x \in S_n \), \( k < n \), and \( y \in T^k[B_{d_T}(x, \varepsilon)] \). Pick \( z \in B_{d_T}(x, \varepsilon) \) so that \( T^k(z) = y \). Observe that

\[
d(T^k(x), y) = d(T^k(x), T^k(z)) \leq d_n^{T_d}(x, z) < \varepsilon.
\]
It follows that
\[ T^k[B_d^k(x, \varepsilon)] \subseteq B_d(T(x), \varepsilon) \]
which implies that
\[ \text{diam}_d(T^k[B_d^k(x, \varepsilon)]) \leq 2\varepsilon < \mathcal{L}_d(\mathcal{U}). \]
Since \( x \in S_n \) and \( k < n \) were arbitrary, we can pick \( U(x, k) \in \mathcal{U} \) so that
\[ T^k[B_d^k(x, \varepsilon)] \subseteq U(x, k). \]
It follows that
\[ B_d^k(\varepsilon) \subseteq T^{-k}[U(x, k)] \Rightarrow B_d^k(x, \varepsilon) \subseteq \bigcap \{ T^{-k}[U(x, k)] : k < n \}. \]
This allows us to define \( \phi : S_n \to U_z^n \) by \( B_d(\varepsilon) \subseteq \phi(x) \).
Recall that a maximal \( (n, \varepsilon) \)-separated set is also \( (n, \varepsilon) \)-spanning. Hence, \( \{ \phi(x) : x \in S_n \} \) is a sub-cover of \( \mathcal{U}_z^n \). Recalling Proposition 25, we have that
\[ N(\mathcal{U}_z^n) \leq R_{\text{span}}(X, n, \varepsilon) \leq \log N(\mathcal{U}_z^n) \leq \log N(\varepsilon, d_z^n) \Rightarrow \log N(\mathcal{U}_z^n) \leq \frac{\log N(\varepsilon, d_z^n)}{n}. \]
Finally, we see that
\[ h(T, \mathcal{U}) \leq R_T(\varepsilon, d) \Rightarrow h(T) \leq \lim \epsilon \to 0 R_T(\varepsilon, d), \]
bringing the proof to its completion. \( \square \)

**Definition 30.** Let \( f : X \to Y \) be a function where \( X \) and \( Y \) are metric spaces with compatible metrics \( d \) and \( \rho \), respectively. We say that \( f \) is **Lipschitz continuous** if there exists \( L \in [0, \infty) \) so that, for any \( x, y \in X \),
\[ \rho(f(x), f(y)) \leq L d(x, y). \]
For a Lipschitz continuous map \( f \), let
\[ L_f = \inf \{ L \geq 0 : (\forall x, y \in X)(\rho(f(x), f(y)) \leq L d(x, y)) \}. \]
If \( L_f \leq 1 \), we say that \( f \) is a **contraction**.

**Proposition 31.** Let \( f : X \to Y \) be a Lipschitz continuous map where \( X \) and \( Y \) have compatible metrics \( d \) and \( \rho \), respectively. Then, for all \( x, y \in X \),
\[ \rho(f(x), f(y)) \leq L_f d(x, y). \]

**Definition 32.** Let \( X \) be a compact metric space with compatible metric \( d \) and define \( b_d : (0, \infty) \to \mathbb{N} \) by
\[ b_d(\varepsilon) = \min \left\{ n \in \mathbb{N} : \left( \exists x_1, \ldots, x_n \in X \right) \left( X \subseteq \bigcup_{i=1}^{n} B_d(x_i, \varepsilon) \right) \right\}. \]
Then we define the **ball dimension** of \( X \) to be
\[ \text{bdim}(X, d) = \lim \sup \varepsilon \to 0 \frac{b_d(\varepsilon)}{\log \varepsilon}. \]
Proposition 33. Let \((X, T)\) be a dynamical system with compatible metric \(d\) so that \(\text{bdim}(X, d) < \infty\). Additionally suppose that \(T\) is Lipschitz continuous. If \(T\) is a contraction, then \(h(T) = 0\). Otherwise,
\[
h(T) \leq \log(L_T) \cdot \text{bdim}(X, d).
\]

**Proof.** Let \(\varepsilon > 0\). Notice that, for \(n \in \mathbb{N}\), \(d(T^n x, T^n y) \leq L_T^n d(x, y)\).

First, assume that \(T\) is a contraction, that is, that \(L_T \leq 1\). Witness that \(d(T^n x, T^n y) \leq L_T^n d(x, y) \leq d(x, y)\). Hence, \(d_T^n \leq d\). Suppose
\[
A = \{x_i : i < b_d(\varepsilon)\}
\]
spans \(X\). Let \(x \in X\) be arbitrary and pick \(x_i\) so that \(x \in B_d(x_i, \varepsilon)\). As \(d_T^n(x, x_i) \leq d(x, x_i) < \varepsilon\), we see that \(A\) is \((n, \varepsilon)\)-spanning. Hence,
\[
\limsup_{n \to \infty} \frac{\log R_{\text{span}}(X, n, \varepsilon)}{n} \leq \limsup_{n \to \infty} \frac{\log b_d(\varepsilon)}{n} = 0.
\]
It follows that \(h(T) = 0\).

Now, we assume that \(L_T > 1\). Let \(n \in \mathbb{N}\) and let
\[
A = \{x_i : i < b_d(L_T^{-n} \varepsilon)\}
\]
span \(X\). We wish to show that \(A\) is \((n, \varepsilon)\)-spanning. For \(x \in X\), let \(x_i \in A\) be so that \(x \in B_d(x_i, L_T^{-n} \varepsilon)\). Observe that
\[
d_T^n(x, x_i) \leq L_T^n d(x, x_i) < \varepsilon.
\]
Hence, \(A\) is \((n, \varepsilon)\)-spanning. It follows that
\[
\frac{\log R_{\text{span}}(X, n, \varepsilon)}{n} \leq \frac{\log b_d(L_T^{-n} \varepsilon)}{n} \leq \frac{b_d(L_T^{-n} \varepsilon)}{|\log(L_T^{-n} \varepsilon)|} \frac{\log(L_T^{-n} \varepsilon)}{n}.
\]

Note that
\[
\limsup_{n \to \infty} \frac{|\log(L_T^{-n} \varepsilon)|}{n} = \limsup_{n \to \infty} \frac{|\log(\varepsilon) - n \log(L_T)|}{n} \leq \limsup_{n \to \infty} \left( \frac{|\log(\varepsilon)|}{n} + \frac{n \log(L_T)}{n} \right) = \log(L_T).
\]

Since \(L_T^{-n} \varepsilon \to 0\) as \(n \to \infty\), we obtain that
\[
\limsup_{n \to \infty} \frac{\log R_{\text{span}}(X, n, \varepsilon)}{n} \leq \text{bdim}(X, d) \log(L_T).
\]
Therefore, taking \(\varepsilon \to 0\), we see that \(h(T) \leq \log(L_T) \cdot \text{bdim}(X, d)\). \(\square\)

Recall the definition

**Definition 34.** Let \(f : X \to Y\) where \(X\) and \(Y\) are metric spaces with compatible metrics \(d\) and \(\rho\), respectively. We say that \(f\) is Hölder continuous if there exists \(k \in [0, \infty)\) and \(\alpha \in [0, \infty)\) so that
\[
\rho(f(x), f(y)) \leq kd(x, y)^\alpha.
\]
Observe that $\beta > 0$ be so that $\forall x, \beta \in G$.

Let compact set consisting of wandering points. Appealing to Proposition 10, Proof. Let $d$ be a compatible metric on $X$. Fix $n \in \mathbb{N}_1$ and $\varepsilon > 0$ and consider the set $S_{\text{span}}(\Omega, n, \varepsilon)$. Let

$$U = \{ x \in X : (\exists y \in S_{\text{span}}(\Omega, n, \varepsilon))(d_n^T(x, y) < \varepsilon) \}.$$ 

Observe that $U$ is open and that $\Omega \subseteq U$. It follows that $K := X \setminus U$ is a compact set consisting of wandering points. Appealing to Proposition 10, let $\beta > 0$ be so that $\beta \subseteq \varepsilon$ and

$$(\forall x \in K)(\forall k \in \mathbb{N}_1)(T^k[B(x, \beta)] \cap B(x, \beta) = \emptyset).$$

Consider the set $S_{\text{span}}(K, n, \beta)$ and let

$$G(n) = S_{\text{span}}(\Omega, n, \varepsilon) \cup S_{\text{span}}(K, n, \beta).$$

Observe that $G(n)$ is an $(n, \varepsilon)$-spanning set for $X$. It follows that

$$R_{\text{span}}(X, n, \varepsilon) \leq \#G(n).$$

Now we’ll define a map $\phi : X \to G(n)^\mathbb{N}$. Fix $x \in X$ and let $k \in \mathbb{N}$. We consider two cases.

- If $T^{kn}(x) \in U$, let $\phi(x)_k \in S_{\text{span}}(\Omega, n, \varepsilon)$ be so that

$$d_n^T(T^{kn}(x), \phi(x)_k) < \varepsilon.$$ 

This choice can be made by our definition of $U$.
- If $T^{kn}(x) \in K$, let $\phi(x)_k \in S_{\text{span}}(K, n, \beta)$ be so that

$$d_n^T(T^{kn}(x), \phi(x)_k) < \beta.$$ 

This choice can be made by definition of $S_{\text{span}}(K, n, \beta)$.

So $\phi(x) = \langle \phi(x)_k : k \in \mathbb{N} \rangle$. We define, for $\ell \in \mathbb{N}_1$, $\phi_\ell : X \to G(n)^\ell$ by the rule $\phi_\ell(x) = \langle \phi(x)_0, \phi(x)_1, \ldots, \phi(x)_\ell-1 \rangle$.

**Claim 35.1.** Suppose $\phi(x)_j, \phi(x)_k \in S_{\text{span}}(K, n, \beta)$ and $\phi(x)_j = \phi(x)_k$. Then $k = j$.

**Reason.** Suppose $k \neq j$ and, without loss of generality, let $j < k$. Let $m$ be so that $j + m = k$ and observe that $jn + mn = kn$. Notice that $T^{jn}(x) \in B(\phi(x)_j, \beta)$ and that

$$T^{mn}[B(\phi(x)_j, \beta)] \cap B(\phi(x)_j, \beta) = \emptyset.$$ 

It follows that $d(T^{kn}(x), \phi(x)_j) \geq \beta$. Since

$$d(T^{kn}(x), \phi(x)_k) \leq d_n^T(T^{kn}(x), \phi(x)_k) < \beta,$$

we see that $\phi(x)_k \neq \phi(x)_j$. ◊
From this claim, we see that
\[ \# \{ (k, \phi(x)_k) : \phi(x)_k \in S_{\text{span}}(K, n, \beta) \} \leq \# S_{\text{span}}(K, n, \beta) = R_{\text{span}}(K, n, \beta). \]

Let \( N > nR_{\text{span}}(K, n, \beta) \) and consider \( S_{\text{sep}}(X, N, 2\varepsilon) \). Then, pick \( \ell \) so that \((\ell - 1)n < N \leq \ell n\).

**Claim 35.2.** For any \( k \geq \ell \), \( \phi_k \mid_{S_{\text{sep}}(X, N, 2\varepsilon)} \) is injective. Moreover, \( \phi \mid_{S_{\text{sep}}(X, N, 2\varepsilon)} \) is injective.

**Reason.** Let \( x, y \in S_{\text{sep}}(X, N, 2\varepsilon) \) be so that \( \phi(x) = \phi(y) \). For \( t < n \) and \( s < \ell \), notice that
\[
  d(T^{sn+t}(x), T^{sn+t}(y)) \leq d(T^{sn+t}(x), T^t(\phi(x)_s)) + d(T^t(\phi(y)_s), T^{sn+t}(y))
  \leq d^t_n(T^s(x), \phi(x)_s) + d^t_n(\phi(y)_s, T^s(y))
  < 2\varepsilon.
\]
as \( N \leq \ell n \), we see that
\[ d^\ell_N(x, y) < 2\varepsilon \]
and, as \( S_{\text{sep}}(X, N, 2\varepsilon) \) is \((N, 2\varepsilon)\)-separated, \( x = y \). \( \diamondsuit \)

**Claim 35.3.** Let \( Q = R_{\text{span}}(K, n, \beta) \) and \( P = R_{\text{span}}(\Omega, n, \varepsilon) \). Then
\[ \# S_{\text{sep}}(X, N, 2\varepsilon) = \# \phi_{\ell}[S_{\text{sep}}(X, N, 2\varepsilon)] \leq (Q + 1)! e^Q P^\ell. \]

**Reason.** Let
\[ A_j = \{ \phi(x) : x \in S_{\text{sep}}(X, N, 2\varepsilon), \#(\{ \phi(x)_k : k < \ell \} \cap S_{\text{span}}(K, n, \beta)) = j \}. \]
Notice that \( j \leq \# S_{\text{span}}(K, n, \beta) = R_{\text{span}}(K, n, \beta) = Q \). We see that \( Q \leq \ell \) as
\[ nQ = nR_{\text{span}}(X, n, \beta) < N \leq n\ell. \]
Hence, \( \phi_{\ell}[S_{\text{sep}}(X, N, 2\varepsilon)] = \bigcup \{ A_j : j \leq Q \} \).

By Claim 35.1, it is clear that
\[ A_j \subseteq B_j := \{ \bar{x} \in G(n)^\ell : \# \{ k < \ell : \bar{x}_k \in S_{\text{span}}(K, n, \beta) \} = j \}. \]
Notice that
\[ \# B \subseteq S_{\text{span}}(K, n, \beta) : \# B = j = \frac{Q!}{j!(Q - j)!}. \]
Observe that there are \( \frac{\ell!}{(\ell - j)!} \) many ways to arrange \( j \) objects into an \( \ell \)-tuple. Now, the rest of the places in the \( \ell \)-tuple are filled with members of \( S_{\text{span}}(\Omega, n, \varepsilon) \) and there are \( P^{\ell - j} \) ways to do that. Thus,
\[ \# A_j \leq \# B_j \leq \frac{Q!}{j!(Q - j)!} \cdot \frac{\ell!}{(\ell - j)!} \cdot P^{\ell - j} \leq Q! \ell! P^\ell \leq Q! \ell^\ell Q P^\ell. \]
As the family \( \{B_j : j \leq Q\} \) is pair-wise disjoint, we have that \( \{A_j : j \leq Q\} \) is pair-wise disjoint and that

\[
\#\phi_\ell[S_{\text{sep}}(X, N, 2\varepsilon)] = \sum_{j=0}^{Q} \#A_j \leq \sum_{j=0}^{Q} \#B_j \leq (Q + 1)!\ell^Q P^\ell,
\]

establishing the claim. ♦

By the previous claim, we have that

\[
R_{\text{sep}}(X, N, 2\varepsilon) \leq (Q + 1)!\ell^Q P^\ell.
\]

So

\[
\frac{\log R_{\text{sep}}(X, N, 2\varepsilon)}{N} \leq \frac{\log((Q + 1)!\ell^Q P^\ell)}{(\ell - 1)n} = \frac{\log((Q + 1)! + Q \log(\ell) + \ell \log(P))}{(\ell - 1)n}.
\]

Since \( N \) and \( \ell \) were arbitrary, we see that

\[
\limsup_{N \to \infty} \frac{\log R_{\text{sep}}(X, N, 2\varepsilon)}{N} \leq \lim_{\ell \to \infty} \frac{\log((Q + 1)! + Q \log(\ell) + \ell \log(P))}{(\ell - 1)n} \leq \frac{\log P}{n}.
\]

That is,

\[
H_{\text{sep}}(X, 2\varepsilon) \leq \frac{\log R_{\text{span}}(\Omega, n, \varepsilon)}{n}.
\]

Therefore,

\[
H_{\text{sep}}(X, 2\varepsilon) \leq \limsup_{n \to \infty} \frac{\log R_{\text{span}}(\Omega, n, \varepsilon)}{n} = H_{\text{span}}(\Omega, \varepsilon).
\]

Therefore, taking \( \varepsilon \to 0 \), we see that

\[
h(X, T) \leq h(\Omega, T |\Omega).
\]

3. MEAN TOPOLOGICAL DIMENSION

The following notion of mean topological dimension was suggested by Gromov but was first studied in depth by Elon Lindenstrauss and Benjamin Weiss in [2].

**Definition 36.** Let \((X, T)\) be a dynamical system and \(\mathcal{U}\) be a finite open cover of \(X\). Then, let

\[
\text{mdim}(X, T, \mathcal{U}) = \lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U}_T^n)}{n}.
\]

Then

\[
\text{mdim}(X, T) := \sup\{\text{mdim}(X, T, \mathcal{U}) : \mathcal{U} \text{ is a finite open cover of } X\}.
\]

Finally,

\[
\text{mdim}(X) := \sup\{\text{mdim}(X, T) : T \in \mathcal{C}(X, X)\}.
\]

**Remark.** It was shown in [2] that \(\mathcal{D}\) behaves sub-additively so the limit exists.
The mean dimension acts somewhat like entropy for the following reason.

**Proposition 37.** For a dynamical system \((X, T)\) and \(k \in \mathbb{N}\),
\[
\text{mdim}(X, T^k) = k \cdot \text{mdim}(X, T).
\]

**Proposition 38.** Let \(X\) be so that \(\dim X < \infty\). Then \(\text{mdim}(X) = 0\).

**Proof.** Let \(T \in \mathcal{C}(X, X)\) and \(\mathcal{U}\) be any finite open cover of \(X\). Notice that
\[
\lim_{n \to \infty} \frac{\mathcal{D}(\mathcal{U}_n)}{n} \leq \lim_{n \to \infty} \frac{\dim X}{n} = 0.
\]
Since \(T\) was arbitrary, this guarantees that \(\text{mdim}(X) = 0\).

**Notation.** Let \(\mathcal{A}\) be a family of subsets of \(X\) and \(E \subseteq X\). Then we let \(\mathcal{A} \upharpoonright E = \{A \cap E : A \in \mathcal{A}\}\).

**Proposition 39.** Let \((X, T)\) be a dynamical system and \(F \subseteq X\) be a closed \(T\)-invariant subset. Then \(\text{mdim}(F, T \upharpoonright F) \leq \text{mdim}(X, T)\).

**Proof.** Let \(\mathcal{U}\) be any finite open cover of \(F\), \(W\) be an open set so that \(F \subseteq W\) and let \(V = \{X \setminus F\} \cup \{U \cup (W \setminus F) : U \in \mathcal{U}\}\).

For sure \(V\) covers \(X\) so we need only check that each \(U \cup (W \setminus F)\) is open to establish that \(V\) is a finite open cover of \(X\). As \(U\) is relatively open with respect to \(F\), let \(V\) be open in \(X\) so that \(V \cap F = U\). Notice that \(U = V \cap F = (W \cap V) \cap F\). Since \((W \cap V) \setminus F \subseteq W \setminus F\), we see that \(W \cap V \subseteq U \cup (W \setminus F)\). Hence, for any \(x \in U, x \in W \cap V\), an open set. That is, \(V\) is an open cover of \(X\).

Now, let \(n \geq 1\) and choose a finite open cover \(\mathcal{W}\) of \(X\) that refines \(\mathcal{V}_n\) so that \(\mathcal{D}(\mathcal{V}_{n}) = \text{ord}(\mathcal{W})\). Note that \(\text{ord}(\mathcal{W} \upharpoonright F) \leq \text{ord}(\mathcal{W})\) and check that \(\mathcal{W} \upharpoonright F\) refines 
\[
\mathcal{A}_n := (T \upharpoonright F)^{-(n-1)}[\mathcal{W}] \vee \cdots \vee (T \upharpoonright F)^{-1}[\mathcal{W}] \vee \mathcal{W}.
\]

It follows that \(\mathcal{D}(\mathcal{A}_n) \leq \text{ord}(\mathcal{W}) = \mathcal{D}(\mathcal{V}_{n})\). Dividing by \(n\), taking the appropriate limit and supremum, we see that \(\text{mdim}(F, T \upharpoonright F) \leq \text{mdim}(X, T)\).

**Lemma 40.** Let \(\mathcal{A}\) be any family of subsets of \(X\). Then
\[
\text{ord}(T^{-1}[\mathcal{A}]) \leq \text{ord}(\mathcal{A}).
\]

**Proof.** Let \(\mathcal{A} = \{A_i : i < \# \mathcal{A}\}\) and notice that
\[
\sum_{A \in \mathcal{A}} \mathbb{1}_A(x) = \#\{i : x \in A_i\}
\]
for any \(x \in X\). Let 
\[
\mathcal{B} = \{B_i := T^{-1}[A_i] : i < \# \mathcal{A}\} = T^{-1}[\mathcal{A}]
\]
and let \(x\) be so that
\[
\sum_{B \in \mathcal{B}} \mathbb{1}_B(x) = \max_{y \in X} \sum_{B \in \mathcal{B}} \mathbb{1}_B(y).
\]
Notice that, if \( x \in B_i \), \( T(x) \in A_i \). That is,
\[
\text{ord} \left( T^{-1} [\mathcal{A}] \right) = -1 + \# \{ i : x \in B_i \} \leq -1 + \# \{ i : T(x) \in A_i \} \leq \text{ord} \left( \mathcal{A} \right).
\]

\[\square\]

**Proposition 41.** For a dynamical system \((X, T)\),
\[
\text{mdim} \left( X, T \right) \preceq \text{mdim} \left( [X], T \upharpoonright_{[X]} \right).
\]

**Proof.** Let \( F = [X], T_F = T \upharpoonright_{[X]} \). Notice that, for any \( A \subseteq X \), \( T^{-1}[A] = T^{-1}[A \cap F] \). It follows that for a cover \( \mathcal{U} \) of \( X \) and \( n \geq 1 \),
\[
T^{-n}[\mathcal{U}] = T^{-n}[\mathcal{U} \cap F].
\]

Let \( n \geq 1 \) and suppose \( \mathcal{V} \) is a cover of \( F \) refining \( \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \) so that
\[
\text{ord} \left( \mathcal{V} \right) = D \left( \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \right).
\]

Notice that, as \( \mathcal{V} \) refines \( \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \), we have that \( T^{-1}[\mathcal{V}] \) refines \( T^{-1} \left[ \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \right] = \bigvee_{i=1}^{n} T^{-i}[\mathcal{U} \cap F] \).

Then, note that, using the sub-additivity of \( D \),
\[
D \left( \bigvee_{i=0}^{n} T^{-i}[\mathcal{U}] \right) \leq D \left( \bigvee_{i=1}^{n} T^{-i}[\mathcal{U}] \right) + D(\mathcal{U}) \leq \text{ord} \left( T^{-1}[\mathcal{V}] \right) + D(\mathcal{U})
\]
\[
\leq \text{ord} \left( \mathcal{V} \right) + D(\mathcal{U}) = D \left( \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \right) + D(\mathcal{U}).
\]

Hence,
\[
\frac{1}{n+1} \cdot D \left( \bigvee_{i=0}^{n} T^{-i}[\mathcal{U}] \right) \leq \frac{1}{n} \cdot D \left( \bigvee_{i=0}^{n-1} T^{-i}[\mathcal{U} \cap F] \right) + \frac{1}{n} \cdot D(\mathcal{U}).
\]

Then, taking \( n \to \infty \) and supremums over finite open covers, we see the desired result. \( \square \)

### 4. Metric Mean Dimension

**Definition 42.** Let \((X, T)\) be a dynamical system and \(d\) be a compatible metric on \(X\). We define the **metric mean dimension** of \((X, T, d)\) as follows:
\[
\text{mdim}_M (X, T, d) = \lim \inf_{\varepsilon \to 0} \frac{R_T(\varepsilon, d)}{\log \varepsilon}.
\]

For a topological invariant, we define
\[
\text{mdim}_M (X, T, d) = \inf \{ \text{mdim}_M (X, T, d) : d \text{ is a compatible metric} \}.
\]

**Proposition 43.** Let \((X, T)\) be a dynamical system so that \(h(T) < \infty\). Then, for any compatible metric \(d\), \(\text{mdim}_M (X, T, d) = 0\).
Proof. As
\[ h(T) = \lim_{\varepsilon \to 0} R_T(\varepsilon, d) = \sup \{ R_T(\varepsilon, d) : \varepsilon \geq 0 \} \]
and \( |\log \varepsilon| \to \infty \) as \( \varepsilon \to 0 \), we see that
\[ \text{mdim}_M(X, T, d) = \lim_{\varepsilon \to 0} R_T(\varepsilon, d) / |\log \varepsilon| \leq \lim_{\varepsilon \to 0} h(T) / |\log \varepsilon| = 0, \]
the promised equality. \( \square \)

Theorem 44. Let \((X, T)\) be a dynamical system and \(\Omega\) be the non-wandering points. Let \(d\) be a compatible metric on \(X\). Then
\[ \text{mdim}_M(X, T, d) \leq \text{mdim}_M(\Omega, T \mid \Omega, d). \]

Moreover,
\[ \text{mdim}_M(X, T) \leq \text{mdim}_M(\Omega \mid \Omega). \]

Proof. Fix \(n \in \mathbb{N}_1\) and \(\varepsilon > 0\) and, for convenience, let \(T^* = T \mid \Omega\). Let
\[ U = \{ x \in X : (\exists y \in S_{\text{span}}(\Omega, n, \varepsilon))(d_n^T(x, y) < \varepsilon) \}. \]
Observe that \(U\) is open and, since \(S_{\text{span}}(\Omega, n, \varepsilon)\) is an \((n, \varepsilon)\)-spanning set for \(\Omega\), \(\Omega \subseteq U\). It follows that \(K := X \setminus U\) is a compact set consisting of wandering points. Appealing to Proposition 10, let \(\beta > 0\) be so that \(\beta \leq \varepsilon\) and, for each \(x \in K\) and all \(k \in \mathbb{N}_1\), \(T^k[B(x, \beta)] \cap B(x, \beta) = \emptyset\).
Let \(N > nR_{\text{span}}(K, n, \beta)\). Then, pick \(\ell\) so that \((\ell - 1)n < N \leq n\). Let \(Q = R_{\text{span}}(K, n, \beta)\) and \(P = R_{\text{span}}(\Omega, n, \varepsilon)\) and recall from Theorem 35 that
\[ R_{\text{sep}}(X, N, 2\varepsilon) \leq (Q + 1)!^{ \ell Q P^\ell}. \]
Combining Propositions 21 and 25, we obtain that
\[ \mathcal{N}(6\varepsilon, d_n^T) \leq R_{\text{span}}(X, N, 2\varepsilon) \leq R_{\text{sep}}(X, N, 2\varepsilon) \leq (Q + 1)!^{ \ell Q P^\ell}. \]
Then
\[ \frac{\log \mathcal{N}(6\varepsilon, d_n^T)}{N} \leq \frac{\log((Q + 1)!^{ \ell Q P^\ell})}{(\ell - 1)n} = \frac{\log((Q + 1)!)}{(\ell - 1)n} + \ell Q \log(\ell) + \ell \log(P). \]
Taking \(N \to \infty\) and \(\ell \to \infty\), we obtain that
\[ R_T(6\varepsilon, d) \leq \frac{\log P}{n} = \frac{\log R_{\text{span}}(\Omega, n, \varepsilon)}{n} \leq \frac{\log \mathcal{N}(\varepsilon, d_n^T)}{n}. \]
Taking \(n \to \infty\),
\[ R_T(6\varepsilon, d) \leq R_T^*(\varepsilon, d). \]
It follows that
\[ \frac{R_T(6\varepsilon, d)}{|\log(6\varepsilon)|} \cdot \frac{|\log(6\varepsilon)|}{|\log \varepsilon|} = \frac{R_T(6\varepsilon, d)}{|\log \varepsilon|} \leq \frac{R_T^*(\varepsilon, d)}{|\log \varepsilon|} \]
which, after taking \(\lim_{\varepsilon \to 0}\), provides that
\[ \text{mdim}_M(X, T, d) \leq \text{mdim}_M(\Omega \mid \Omega, d), \]
the desired result.
Since the above inequality holds for any compatible metric $d$, we see that

$$\text{mdim}_M(X, T) \leq \text{mdim}_M(\Omega, T|_{\Omega}),$$

ending the proof.

\[\square\]

**References**
