Definition 1. Let $G$ be a group endowed with a topology. We say that $G$ is a semitopological group if, for all $y \in G$, the maps $x \mapsto xy$, $x \mapsto yx$, $G \leftarrow$ are continuous. We say that $G$ is a paratopological group if the map $(x, y) \mapsto xy$, $G^2 \rightarrow G$ is continuous.

Proposition 2. Let $G$ be a paratopological group. Then, for any neighborhood $U$ of 1, there exists another neighborhood $V$ of 1 so that $V^2 \subseteq U$.

Proof. As multiplication is continuous, we can find open sets $A$ and $B$, both containing 1, so that $AB \subseteq U$. Let $V = A \cap B$. Note that $V$ is the desired neighborhood of 1.

Proposition 3. Let $G$ be a semitopological group. Then $G$ is a paratopological group if and only if $(x, y) \mapsto xy$ is continuous at $(1, 1)$.

Proof. Suppose $(x, y) \mapsto xy$ is continuous at $(1, 1)$. Let $\mathcal{F}$ and $\mathcal{G}$ be filters so that $\mathcal{F} \rightarrow a$ and $\mathcal{G} \rightarrow b$. Then $a^{-1}\mathcal{F} \rightarrow 1$ and $\mathcal{G}b^{-1} \rightarrow 1$. By our assumption, $a^{-1}\mathcal{F}\mathcal{G}b^{-1} \rightarrow 1$. It follows that $\mathcal{F}\mathcal{G} \rightarrow ab$, finishing the proof.

Proposition 4. Let $G$ be a paratopological group. Then, $G$ is a topological group if and only if $x \mapsto x^{-1}$ is continuous at 1.

Proof. Assume that $x \mapsto x^{-1}$ is continuous at 1 and let $\mathcal{F}$ be a filter on $G$. Then notice that, for arbitrary $g \in G$,

$\mathcal{F} \rightarrow g \Rightarrow \mathcal{F}g^{-1} \rightarrow 1$
$\Rightarrow g\mathcal{F}^{-1} \rightarrow 1$
$\Rightarrow \mathcal{F}^{-1} \rightarrow g^{-1}$.

That is, $x \mapsto x^{-1}$ is continuous and $G$ is a topological group.

1 Polish case

The arguments in this section follow [4].

Lemma 5. Let $G$ be a complete metrizable semitopological group with metric $d$. Then, for any open set $U \subseteq G$ and any $\varepsilon > 0$, there exists an open set $V \subseteq U$ and a $\delta > 0$ so that, for any $h \in V$ and $a \in G$, $d(a, 1) < \delta$ implies $d(ah, h) \leq \varepsilon$. 

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Proof. For \( n \in \mathbb{Z}^+ \), let \( B_n = \{ h \in G : \forall a \in G \ [d(a, 1) < \frac{1}{n} \Rightarrow d(ah, h) \leq \varepsilon] \} \) and let \( \mathcal{B} = \{ B_n : n \in \mathbb{Z}^+ \} \). To see that \( B_n \) is closed, let \( b \in G \) and \( \{ b_m : m \in \omega \} \subseteq B_n \) be so that \( b_m \to b \). As \( G \) is a semitopological group, \( ab_m \to ab \). Notice that

\[
d(ab, b) \leq d(ab, ab_m) + d(ab_m, b_m) + d(b_m, b).
\]

As this inequality holds for all \( m \), we see that \( d(ab, b) \leq \varepsilon \), establishing that \( b \in B_n \). Hence, \( B_n \) is closed.

We’ll now show that \( \mathcal{B} \) covers \( G \). Let \( g \in G \) be arbitrary and let \( \mathcal{N}_1 = \{ A : 1 \in \text{int}(A) \} \), the filter of neighborhoods of 1. As \( G \) is a semitopological group, \( \mathcal{N}_1g \to g \). Since \( B(g, \varepsilon) \) is a neighborhood of \( g \), \( B(g, \varepsilon) \in \mathcal{N}_1g \). Pick \( n \) so large\(^1\) that \( B(1, \frac{1}{n}) \subseteq B(g, \varepsilon)g^{-1} \) which implies that \( B(1, \frac{1}{n})g \subseteq B(g, \varepsilon) \). Let \( a \) be so that \( d(a, 1) < \frac{1}{n} \). It follows that \( ag \in B(g, \varepsilon) \) which means that \( d(ag, g) < \varepsilon \). So, for this particular choice of \( n, g \in B_n \). That is, \( \mathcal{B} \) is covers \( G \).

Finally, let \( U \) be any open subset of \( G \). As \( G \) is complete metrizable, \( U \) is non-meager and there exists some \( n \) so that \( U \cap B_n \) is non-meager. In particular, \( B_n \) is non-meager so has a non-empty interior. If \( U \cap B_n \subseteq \partial B_n \), \( U \cap B_n \) would be nowhere dense, a contradiction. Hence, it must be the case that \( U \cap \text{int}(B_n) \) is a non-empty open set. This finishes the proof. \( \square \)

**Theorem 6.** If \( G \) is a complete metrizable semitopological group, \( G \) is a paratopological group.

*Proof.* By Proposition 3, it suffices to show that \( (x, y) \mapsto xy \) is continuous at \((1,1)\).

Let \( U_0 = G \). Let \( V \) be an open set in \( G \) with \( \text{diam}(V) < 1 \). Apply the previous lemma to obtain an open \( U_1 \subseteq V \) and \( \delta_1 > 0 \) so that

\[
\forall h \in U_1 \forall a \in G \ (d(a, 1) < \delta_1 \implies d(ah, h) \leq 1).
\]

By way of induction, suppose, for \( n \geq 1 \), for all \( k \leq n - 1 \), we have a number \( \delta_k > 0 \) and \( U_k \) open so that \( \text{diam}(U_k) < \frac{1}{k} \),

\[
\forall h \in U_k \forall a \in G \left( d(a, 1) < \delta_k \implies d(ah, h) \leq \frac{1}{k} \right),
\]

and \( \text{cl}(U_k) \subseteq U_{k-1} \). Let \( W \subseteq U_{n-1} \) be open so that \( \text{diam}(W) < \frac{1}{n} \) and \( \text{cl}(W) \subseteq U_{n-1} \). Now, apply the lemma to obtain an open set \( U_n \) with \( U_n \subseteq W \) and

\[
\forall h \in U_n \forall a \in G \left( d(a, 1) < \delta_n \implies d(ah, h) \leq \frac{1}{n} \right).
\]

Now, \( \{ U_n : n \in \omega \} \) is a sequence of open sets with \( \text{cl}(U_{n+1}) \subseteq U_n \) and \( \text{diam}(U_n) \to 0 \) as \( n \to \infty \). Hence, let \( g \in G \) be so that \( \{ g \} = \bigcap \{ U_n : n \in \omega \} \).

Let \( \varepsilon > 0 \) be arbitrary. As \( x \mapsto xg^{-1} \) is continuous, it is, in particular, continuous at \( g \). So let \( \delta > 0 \) be so that \( d(x, g) < \delta \) implies \( d(xg^{-1}, gg^{-1}) = d(xg^{-1}, 1) < \varepsilon \). Let \( n \) be big enough so that \( \frac{1}{n} < \delta \) and let \( U = B(1, \delta_2n) \cap (U_{2n}g^{-1}) \). Observe that \( U \) is a neighborhood of 1 so \( U^2 \) is a neighborhood of \((1,1)\). Let \( (a, b) \in U^2 \). Let \( b_0 \in U_{2n} \) so that \( b = b_0g^{-1} \). Observe that, as \( a \in B(1, \delta_2n) \) and \( \text{diam}(U_{2n}) < \frac{1}{2n} \),

\[
d(ab_0, g) \leq d(ab_0, b_0) + d(b_0, g) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} < \delta.
\]

It follows that

\[
d(ab, 1) = d(ab_0g^{-1}, 1) < \varepsilon.
\]

As \( a \) and \( b \) were arbitrary, the proof is finished. \( \square \)

\(^1\)as \( B(g, \varepsilon)g^{-1} \) is a neighborhood of 1.
Lemma 7. If $G$ is a Polish paratopological group and $x \mapsto x^{-1}$ is Baire-measurable, $x \mapsto x^{-1}$ is continuous.

Proof. By Proposition 4, it suffices to show that $x \mapsto x^{-1}$ is continuous at 1. As $x \mapsto x^{-1}$ is Baire-measurable, there is a set $A \subseteq G$ so that $[x \mapsto x^{-1}] |_A$ is continuous and $A$ is co-meager in $G$. Let $a_n \to 1$ and notice that $a_n^{-1}A$ is co-meager for each $n$. It follows that

$$B = A \cap \bigcap \{a_n^{-1}A : n \in \omega\}$$

is co-meager in $G$. In particular, $B$ is non-empty. So, let $g \in B$. Then, $a_n g \in A$ for each $n$. As $[x \mapsto x^{-1}] |_A$ is continuous $a_n g \to g$, $g^{-1}a_n^{-1} \to g^{-1}$. As translation is continuous, we see that $a_n^{-1} \to 1$. Hence, $x \mapsto x^{-1}$ is continuous at 1, finishing the proof of the lemma. □

Theorem 8. If $G$ is a Polish semitopological group, $G$ is a topological group.

Proof. From Theorem 6, $G$ is a paratopological group. Hence, by Proposition 4, we need only show $x \mapsto x^{-1}$ is continuous at 1. To accomplish that, Lemma 7 guarantees that it suffices to check that $x \mapsto x^{-1}$ is Baire-measurable. We will actually show that $x \mapsto x^{-1}$ is Borel-measurable.

Let $F \subseteq G$ be a closed set and $A = \{(x, y) \in G^2 : xy = 1\}$. $A$ is closed in $G^2$ as multiplication is continuous. Consider the set $F^* = (G \times F) \cap A$, a closed subset of $G^2$, and let $\pi : G^2 \to G$ be the canonical projection onto the first coordinate. Observe that

$$x \in \pi(F^*) \iff \exists z \in F^* \ x = \pi(z) \iff \exists y \in F \ xy = 1 \iff x \in F^{-1}.$$

Hence, $\pi(F^*) = F^{-1}$. Immediately, $F^{-1} \subseteq \Sigma^1_1$. To see that $F^{-1}$ is Borel, we will show that $\pi |_{F^*}$ is an injection. As $F^* \subseteq A$, it suffices to check that $\pi |_{A}$ is an injection. This follows from uniqueness of the inverse. Whereby, for any closed $F$, $F^{-1}$ is Borel, $x \mapsto x^{-1}$ is a Borel mapping, finishing the proof. □

2 Locally compact Hausdorff case

The arguments in this section follow [3].

Let $G$ be a locally compact Hausdorff paratopological group throughout.

Lemma 9. If $A \subseteq G$ is compact, $A^{-1}$ is closed.

Proof. Let $x \in \text{cl}(A^{-1})$ and $\mathcal{F}$ be an ultrafilter on $G$ so that $\mathcal{F} \to x$. As $A$ is compact, it follows that $\mathcal{F}^{-1}$ has a limit point $y \in A$. As multiplication is continuous, we get that $\mathcal{F} \mathcal{F}^{-1} \to xy$. As $G$ is Hausdorff, $xy = 1$ so $y^{-1} = x \in A^{-1}$. Hence, $A^{-1}$ is closed. □

Lemma 10. Let $E \subseteq G$ be countable and suppose $x \in \overline{E}$. Then $x^{-1} \in \text{cl}(E^{-1})$.

Proof. Suppose $x \in \overline{E}$, let $E_* = E \cup \{x\}$, and $H = \bigcup \{E^n : n \in \mathbb{Z}\}$. Also, let $\mathcal{F}$ be a filter (on $E$) so that $\mathcal{F} \to x$. $H$ is a countable subgroup of $G$. Notice that $H^2 \subseteq H$ as $H$ is a subgroup. As multiplication is continuous, $\overline{H^2} \subseteq \text{cl}(H^2) \subseteq \overline{H}$.

Let $V$ be a compact neighborhood of 1. Let $g \in \overline{H}$ and, as $H$ is countable, let $\{g_n : n \in \omega\} \subseteq H$ be so that $g_n \to g$. Then $g^{-1}g_n \to 1$ as multiplication is continuous. There is an $N$ big enough so that $g^{-1}g_N \in V$. Then $g_N^{-1}g \in V^{-1}$. It follows that $g \in g_NV^{-1} \subseteq HV^{-1}$. That is, $\overline{H} \subseteq HV^{-1}$.

Now, we have that

$$\overline{H} = \bigcup \{h(\overline{H} \cap V^{-1}) : h \in H\}.$$
Since $\overline{H}$ is a closed subspace of a locally compact space, $\overline{H}$ is itself locally compact. Since $H$ is countable, there exists an $h_0 \in H$ so that $h_0 (\overline{H} \cap V^{-1})$ is non-meager in $\overline{H}$. Moreover, as $h_0 (\overline{H} \cap V^{-1})$ is closed ($V^{-1}$ is closed by Lemma 9), $\text{int}_{\overline{H}} (h_0 (\overline{H} \cap V^{-1})) \neq \emptyset$. So there exists an open subset $U$ of $X$ so that

$$\emptyset \neq U \cap \overline{H} \subseteq h_0 (\overline{H} \cap V^{-1}).$$

There exists $h_1 \in H$ so that $h_1 \in U$. It follows that $h_1^{-1}U$ is a neighborhood of $1$. Hence, $xh_1^{-1}U$ is a neighborhood of $x$. Note that $xh_1^{-1}\overline{H} = \overline{H}$. Then,

$$xh_1^{-1}U \cap \overline{H} = xh_1^{-1} (U \cap \overline{H}) \subseteq xh_1^{-1}h_0 (\overline{H} \cap V^{-1}) \subseteq xh_1^{-1}h_0 V^{-1}.$$

As $\mathcal{F} \to x$, there exists $F \in \mathcal{F}$ so that $F \subseteq xh_1^{-1}h_0 V^{-1}$. Hence, $F^{-1} \subseteq Vh_0^{-1}h_1 x^{-1}$. As $V$ is compact, we see that $\mathcal{F}^{-1}$ converges to some $y \in G$. Using continuity of multiplication, we see that $\mathcal{F} \mathcal{F}^{-1} \to xy$ and $\mathcal{F} \mathcal{F}^{-1} \to 1$. As $G$ is Hausdorff, $y = x^{-1}$ so $\mathcal{F}^{-1} \to x^{-1}$ and the proof is finished. 

**Lemma 11.** If $A \subseteq G$ is compact, $A^{-1}$ is compact.

**Proof.** Let $V$ be a compact neighborhood of $1$ and let $V^o$ be the interior of $V$. Suppose, towards a contradiction, that no finite union of translates of $V$ covers $A^{-1}$. So, we can build a sequence $E_0 = \{x_n : n \in \omega\} \subseteq A$ where

$$x_{n+1}^{-1} \notin \bigcup \{x_i^{-1} V : i \leq n\}.$$

Let $E_n = \{x_\ell : \ell \geq n\}$ and notice that, as $A$ is compact, there exists

$$x \in \bigcap \{E_n : n \in \omega\}.$$

Let $m$ be so that $x_m \in V^o x$. Then $x^{-1} \in x_m^{-1} V^o$. It follows that $x_m^{-1} V$ is a neighborhood of $x^{-1}$. By Lemma 10, $x^{-1} \in \text{cl}(E_m^{-1})$. Then there exists $k \geq m + 1$ so that $x_k^{-1} \in x_m^{-1} V$, a contradiction. Therefore, $A^{-1}$ can be covered by a finite number of translates of $V$. Conclusively, $A^{-1}$ is compact. 

**Theorem 12.** Let $G$ be a locally compact Hausdorff paratopological group. Then $G$ is a topological group.

Actually, more is true. If $G$ is a locally compact Hausdorff semitopological group, $G$ is a topological group. It will not be presented here but can be found in [2].

**Proof.** It suffices to show that $x \mapsto x^{-1}$ is continuous at $1$ by Proposition 4. Let $\mathcal{F}$ be any ultrafilter so that $\mathcal{F} \to 1$. Let $V$ be a compact neighborhood of $1$. Then $V \in \mathcal{F}$. Since $V^{-1}$ is compact by Lemma 11, $\mathcal{F}^{-1}$ converges to some $y \in G$. Since $\mathcal{F} \mathcal{F}^{-1} \to 1$ and multiplication is continuous, $y = 1$. Therefore, $x \to x^{-1}$ is continuous and the proof is complete.

### 3 Ending remarks

**Question 1.** If $G$ is a Čech-complete semitopological group, is $G$ a topological group?

In [1] it is shown that a Čech-complete semitopological group is a topological group if and only if it is paracompact. This answers Question 1.
References


