An Extension of the Baire Property

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Outline

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Preliminaries

**Definition**
By a **Polish space** we mean a separable topological space that admits a compatible complete metric.
Preliminaries

Definition

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Definition

Let $X$ be a topological space. We say that $A \subseteq X$ is **nowhere dense** if $\text{int}(\text{cl}(A)) = \emptyset$ and that it is **meager** if there exists $\{B_n \subseteq X : n \in \mathbb{N}\}$ so that each $B_n$ is nowhere dense and $A \subseteq \bigcup \{B_n : n \in \mathbb{N}\}$. A set $A \subseteq X$ is **co-meager** if $X \setminus A$ is meager.
\[ M(X) \] – the collection of all Borel probability measures on \( X \)

\[ C(X) \] – the collection of continuous functions \( f : X \to \mathbb{R} \)

\( \mu \in M(X) \) given induces a natural map \( f \mapsto \int f \, d\mu \), \( C(X) \to \mathbb{R} \)

\( M(X) \) injects naturally into \( \mathbb{R}^{C(X)} \) so inherits its topology, called the \textbf{weak topology} therefrom
- $\mathcal{M}(X)$ – the collection of all Borel probability measures on $X$
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**Theorem**

*If $X$ is Polish, then $\mathcal{M}(X)$ endowed with the weak topology is a Polish space.*
Definition

A _σ-ideal_ of sets is a collection $I \subseteq \wp(X)$ so that

- if $A \in I$ and $B \subseteq A$, then $B \in I$ and
- for $\{A_n : n \in \mathbb{N}\} \subseteq I$, $\bigcup\{A_n : n \in \mathbb{N}\} \in I$.,
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By construction, the class of meager subsets of a topological space \( X \) is a \( \sigma \)-ideal.
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By construction, the class of meager subsets of a topological space $X$ is a $\sigma$-ideal.

Theorem (Baire Category Theorem)

The $\sigma$-ideal of meager subsets of a Polish space $X$ is proper. That is, $X \notin I$. 

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Extended Baire Prop.
Residually Null Sets

Given $\mu \in \mathcal{M}(X)$, recall that the outer measure $\mu^*$ induced by $\mu$ is a set function $\mu^* : \varnothing(X) \rightarrow [0, 1]$ defined by

$$\mu^*(A) = \inf \{ \mu(U) : U \text{ is open and } A \subseteq U \}.$$

**Definition**

For $A \subseteq X$, we define

$$\mathcal{N}(A) = \{ \mu \in \mathcal{M}(X) : \mu^*(A) = 0 \}.$$

Then we say that $A \subseteq X$ is residually null if $\mathcal{N}(A)$ is co-meager in $\mathcal{M}(X)$. Let $\textbf{RN}(X)$ be the collection of residually null sets.
The following was done by Dubins and Freedman [1] in 1964 for the context when $X$ is compact metrizable.

**Theorem**

Suppose $X$ is a Polish space. If $A \subseteq X$ is meager, $A$ is residually null. On the other hand, if $A$ is a set with the Baire property and is residually null, then $A$ is meager.
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**Theorem**

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- CH, or MA, $\implies$ existence of residually null sets which are non-meager.
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**Theorem**

If \( X \) is a Polish space, then \( \mathcal{M}_c(X) \) is a dense \( G_\delta \) subset of \( \mathcal{M}(X) \).
Definition

For $A \subseteq X$, $A$ is **universally null** if $\mathcal{N}(A) = \mathcal{M}_c(X)$.
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**Definition**

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- universally null $\implies$ residually null
- any perfect meager set is residually null but not universally null
Proposition (Extended BCT)

The collection of residually null subsets of a Polish space $X$ is a $\sigma$-ideal. Moreover, this $\sigma$-ideal is proper, trivially, as $\mu(X) = 1$ for all $\mu \in M(X)$. 

Question

For a Polish space $X$, the collection $I$ of all sets $A$ so that there is a meager set $M$ and a universally null set $U$ with $A = M \cup U$ is a proper $\sigma$-ideal. Then $I \subseteq RN(X)$. Is it the case that $RN(X) = I$?
Proposition (Extended BCT)

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Theorem (C.)

Let $X$ and $Y$ be Polish spaces and suppose that $A \subseteq X$ is a universally null set which is non-meager. Then $A \times Y \in \mathbf{RN}(X \times Y)$ but $A \times Y$ cannot be written as the union of a meager set with a universally null set.
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Let $X$ and $Y$ be Polish spaces and suppose that $A \subseteq X$ is a universally null set which is non-meager. Then $A \times Y \in RN(X \times Y)$ but $A \times Y$ cannot be written as the union of a meager set with a universally null set.

We offer a sketch of the proof.

Lemma

Let $X$ and $Y$ be Polish spaces. Then, for $\varepsilon > 0$, the set

$$\{\mu \in M(X \times Y) : (\exists x \in X)(\mu(\{x\} \times Y) \geq \varepsilon)\}$$

is closed and nowhere dense.
Lemma

Let $X$ and $Y$ be Polish spaces and $A \subseteq X$ be universally null. Then, for $\mu \in \mathcal{M}(X \times Y)$,

$$\mu^*(A \times Y) > 0 \iff (\exists x \in A)(\mu(\{x\} \times Y) > 0).$$
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Lemma

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To finish the argument,

- Let $A$ be universally null and non-meager
- $A \times Y$ is residually null
- By Kuratowski-Ulam Theorem, for any meager set $M$, $(A \times Y) \setminus M$ contains a perfect set of the form $\{x\} \times F$
- $(A \times Y) \setminus M$ is not universally null.
Extended Baire Property

Definition

We define $\text{EBP}(X)$ to be all sets $A$ so that there exists an open set $U$ so that $U \triangle A := (A \cup U) \setminus (A \cap U)$ is residually null. Additionally, given a map $f : X \to Y$ where $X$ and $Y$ are Polish, $f$ is said to be \textbf{EBP-measurable} if $f^{-1}[V] \in \text{EBP}(X)$ for each open set $V \subseteq Y$.

As all meager sets are residually null, the class of $\text{EBP}(X)$ contains all sets with the Baire property.
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Definition

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As all meager sets are residually null, the class of EBP($X$) contains all sets with the Baire property.

If sets which are residually null but non-meager exist, then the class EBP($X$) is a finer class than the class of sets with the Baire property.
In a similar vein, we offer

**Definition**

Let $X$ be a Polish space. We say that a set $A \subseteq X$ is

- **universally measurable** if $A$ is $\mu$-measurable with respect to all continuous measures or
- **residually measurable** if $A$ is $\mu$-measurable with respect to a co-meager collection of measures.

Clearly, all universally measurable sets are residually measurable.

For any closed meager subset $F$, pick some continuous measure $\mu$ on $F$ and let $A \subseteq F$ be a non-$\mu$-measurable set. Then $A$ is residually measurable but not universally measurable.
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Clearly, all universally measurable sets are residually measurable.

For any closed meager subset $F$, pick some continuous measure $\mu$ on $F$ and let $A \subseteq F$ be a non-$\mu$-measurable set. Then $A$ is residually measurable but not universally measurable.
Definition

Let $G$ be an abstract multiplicative group with a Polish topology $\tau$. We say that $G$ (paired with $\tau$) is a **Polish group** provided the maps

- $g \mapsto g^{-1}, G \to G$, and
- $(g, h) \mapsto gh, G^2 \to G$

are continuous.
**Definition**

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**Theorem (C., Kallman)**

Let $G$ and $H$ be Polish groups and suppose $\phi : G \to H$ is a group homomorphism. If $\phi$ is EBP-measurable, $\phi$ is continuous.
\[
\begin{array}{ccc}
\sigma\text{-ideal} & \sigma\text{-algebra} & \text{continuity} \\
\downarrow & \downarrow & \\
\text{meager} & \text{BP} & \checkmark \\
\text{RN} & \text{EBP} & \checkmark \\
\text{UN} & & \\
\uparrow & & \\
\text{UM} & & \\
\downarrow & & \\
\text{RM} & & \\
\end{array}
\]
**Martin’s Axiom**

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Suppose $X$ is a compact Hausdorff space with countable cellularity and let $\text{meag}(X)$ be collection of meager sets. Then $\text{cov}(\text{meag}) \geq c$. 
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MA implies that, for a Polish space $X$ and a cardinal $\kappa < c$,

- $\kappa$-length unions of meager sets are meager
- if $\mu \in \mathcal{M}_c(X)$, then $\kappa$-length unions of $\mu$-null sets are $\mu$-null
If we assume Martin’s Axiom holds, we obtain

**Theorem (C.)**

Suppose $X$ is Polish. For any cardinal $\kappa < \mathfrak{c}$ and any family $F$ of $\kappa$-many residually null sets, $\bigcup F$ is residually null.
Future Work

- $X$ – compact Hausdorff space
- $\mathcal{M}(X)$ – regular Borel probability measures on $X$
- $\mathcal{M}(X)$ is a compact Hausdorff space
- residually null sets, $\mathbf{RN}(X)$ extend naturally here
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Residually Null Martin’s Axiom (RNMA)

Suppose $X$ is a compact Hausdorff space with countable cellularity and let $\text{RN}(X)$ be collection of residually null sets. Then $\text{cov}(\text{RN}) \geq c$. 
1. Is MA equivalent to RNMA?
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2. Does the residually null analog to the Kuratowski-Ulam Theorem hold?
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2. Does the residually null analog to the Kuratowski-Ulam Theorem hold?

3. Inspired by the construction of a residually null set which fails to be the union of a meager set and a universally null set in a space $X \times Y$, can such a set can be constructed in all Polish $X$?
Thank You!