Existence and stability of steady compressible Navier-Stokes solutions on a finite interval with noncharacteristic boundary conditions

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Abstract

We study existence and stability of steady solutions of the isentropic compressible Navier-Stokes equations on a finite interval with noncharacteristic boundary conditions, for general not necessarily small-amplitude data. We show that there exists a unique solution, about which the linearized spatial operator possesses (i) a spectral gap between neutral and growing/decaying modes, and (ii) an even number of nonstable eigenvalues $\lambda$ (with a nonnegative real part). In the case that there are no nonstable eigenvalues, i.e., of spectral stability, we show this solution to be nonlinearly exponentially stable in $H^2 \times H^3$. Using “Goodman-type” weighted energy estimates, we establish spectral stability for small-amplitude data. For large-amplitude data, we obtain high-frequency stability, reducing stability investigations to a bounded frequency regime. On this remaining, bounded-frequency regime, we carry out a numerical Evans function study, with results again indicating universal stability of solutions.

1 Introduction

In this paper, we initiate in the simplest setting of 1D isentropic gas dynamics, a systematic study of existence and stability of steady solutions of systems of hyperbolic parabolic equations on a bounded domain, with noncharacteristic inflow or outflow boundary conditions, and data and solutions of amplitudes that are not necessarily small. We have in mind the scenario of a “shock tube”, or finite-length channel with inflow-outflow boundary conditions, which in turn could be viewed as a generalization of the Poisseuille flow in the incompressible case.

Our conclusions in the present, isentropic case, obtained by rigorous nonlinear and spectral stability theory, augmented in the large-amplitude case by numerical Evans function analysis, are that for any choice of data there exists a unique solution, and this solution is linearly and nonlinearly time-exponentially stable in $H^2 \times H^3$. These results suggest a number of interesting directions for further investigation in 1 and multi-D.

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1.1 Setting

We consider the 1D isentropic compressible Navier-Stokes equations

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x &= \nu u_{xx}
\end{align*}
\]  

on the interval \([0, 1]\), with the noncharacteristic boundary conditions

\[
\begin{align*}
\rho(0) &= \rho_0 > 0, \\
u(0) &= u_0 > 0, \\
u(1) &= u_1 > 0.
\end{align*}
\]  

Notice that we have an inflow boundary condition at \(x = 0\) and an outflow boundary condition at \(x = 1\). We assume that the viscosity \(\nu\) is positive and constant and that the pressure \(P\) is a smooth function satisfying

\[P' > 0.\]

Stability of steady states for hyperbolic parabolic systems has been studied by many authors. For problems on the whole line, the reader can refer to [MZ03, BHRZ08] and references within. In the case of noncharacteristic boundary conditions on the half line, see for instance [CHNZ09, NZ09]. For studies of scalar conservation laws on a bounded interval, one may see for instance [KK86, JP03]. Finally, we refer to [EGGP12] for the study of boundary controllability of the 1D Navier-Stokes equations.

In this paper, we study the existence and stability of steady states of (1) satisfying the boundary conditions (2). Section 2 is devoted to the existence and the uniqueness of such steady states. In Section 3, we study the corresponding linearized problem around the steady state. In section 4, we show that constant steady states and almost constant steady states, see Condition (5), are spectrally stable. We also show that general steady states are numerically spectrally stable. Section 5 is devoted to a local wellposedness result for problem (1)-(2). Then, in section 6, we show the nonlinear stability of steady states that are spectrally stable. Theorem 6.3 is the main result of this paper. Finally, in section 7, we improve the previous theorem under more restrictive assumptions.

Remark 1.1. It is worth noting that boundary conditions (2) are not the only ones we can deal with. For instance, the case

\[
\begin{align*}
\rho(1) &= \rho_0 > 0, \\
u(0) &= u_0 < 0, \\
u(1) &= u_1 < 0,
\end{align*}
\]

is equivalent by the change of variables \(x \rightarrow 1 - x\) and the change of unknowns \((\hat{\rho}, \hat{u}) \rightarrow (\hat{\rho}, -\hat{u})\). Moreover, these two possibilities are the only types of noncharacteristic boundary conditions yielding physically realizable steady states. For, the first equation of (1) yields that steady solutions have constant momentum \(\rho u \equiv m\), so that \(u(0)\) and \(u(1)\) necessarily agree in sign. By similar reasoning, characteristic boundary conditions \(u(0) = u(1) = 0\) yield un \(\equiv 0\) yield only trivial, constant steady states \((\rho, u) \equiv (\rho_0, 0)\).
1.2 Discussion and open problems

As mentioned earlier, our goal in this paper is to open a line of investigation of large-amplitude steady solutions for inflow-outflow problems on bounded domains. The main technical contribution is our argument for nonlinear exponential stability of spectrally stable solutions, which is both particularly simple and also applies to general hyperbolic parabolic systems of “Kawashima type”, as considered on the whole- and half-line in [MZ03, BHRZ08, CHNZ09, NZ09]. Our goal, and the novelty of the argument as compared to those for the whole- and half-line, was to take advantage of the spectral gap to obtain a simple proof based on standard semigroup/energy methods. However, a close reading will reveal that this is deceptively difficult to accomplish, involving the introduction of a precisely chosen space \((\rho, u) \in H^1 \times L^2\) with norm strong enough that we can carry out energy-based high-frequency resolvent estimates and different from the usual Kawashima type estimates, but weak enough that the range of nonlinear terms is densely contained.

The reduction of nonlinear to spectral stability gives a base for investigation of more general systems such as full (nonisentropic) gas dynamics or (isentropic or nonisentropic) MHD. Our results on uniqueness and universal stability on the other hand are likely accidents of low dimension. For example, the demonstration of unstable large-amplitude boundary layers in [SZ01, Zum10] is suggestive via the large-interval length limit from bounded interval toward the half-line, that unstable large-amplitude steady solutions might occur on bounded intervals for polytropic full gas dynamics in some parameter regimes. Definitely, the example of unstable shock waves on the whole line in [BFZ15] together with the asymptotic analysis in [SS00, Zum10] of spectra in the whole-line limit shows that unstable steady solutions are possible on an interval for full gas dynamics with an artificial equation of state satisfying all of the usual requirements imposed in standard theory, including existence of a convex entropy, genuine nonlinearity of acoustic modes, etc.

Moreover, due to the presence of spectral gap/absence of essential spectra in the bounded-interval problem, differently from the whole- and half-line problems, changes in stability of the type considered in [SZ01, Zum10], involving passage of a real eigenvalue through zero, are associated necessarily with bifurcation/nonuniqueness. Thus, any such violations of stability should yield also examples of large-amplitude nonuniqueness at the same time. Small-amplitude uniqueness, on the other hand, follows readily by uniqueness of constant solutions, as follows by energy estimates like those here, plus continuity. The investigation of large-amplitude uniqueness and stability for larger systems thus appears to be a very interesting direction for future exploration; likewise, the study of the corresponding multi-D problem, for which existence/uniqueness of small-amplitude solutions has been studied for example in [KK98, KK97, MP14]. In both 1- and multi-D, a very interesting open problem would be to study the asymptotic structure of solutions in the small-viscosity limit, particularly in the multi-D case analogous to Poiseuille flow.
Notation
In this paper, \( C(\cdot) \) denotes a nondecreasing and positive function and \( C \) a positive constant whose exact values have no importance. \(|\cdot|_2\) refers to the \( L^2 \)-norm on \((0,1)\) and \(|\cdot|_{H^n}, \) for \( n \geq 1,\) to the \( H^n \)-norm. \(|\cdot|_\infty\) refers to the \( L^\infty \)-norm on \([0,1]\).

2 Existence and uniqueness of steady states

2.1 Analytical results

In this part, we prove the following result.

**Proposition 2.1.** For any \((\rho_0, u_0, u_1)\), Problem (1)-(2) has a unique steady solution \((\hat{\rho}, \hat{u})\) with \(\hat{\rho} > 0\).

**Proof.** A steady solution \((\hat{\rho}, \hat{u})\) of (1) with (2) satisfies

\[
\begin{align*}
(\hat{\rho} \hat{u})_x &= 0, \\
(\hat{\rho} \hat{u}^2 + P(\hat{\rho}))_x &= \nu \hat{u}_{xx}, \\
\hat{\rho}(0) &= \rho_0, \\
\hat{u}(0) &= u_0, \quad \hat{u}(1) = u_1.
\end{align*}
\]

Thus,

\[
\begin{align*}
\dot{\rho} \hat{u} &= \rho_0 u_0, \\
\nu \rho_0 u_0 \dot{\rho} &= b \rho^2 - (\rho_0 u_0)^2 \dot{\rho} - \rho^2 P(\dot{\rho}), \\
\dot{\rho}(0) &= \rho_0.
\end{align*}
\]

where \(b\) is a constant that has to be determined. We define the map

\[
\Phi := b \rightarrow u(1) - u_1,
\]

where \((\hat{\rho}, \hat{u})\) is the unique solution of system (4). Notice that we only define this map when \((\hat{\rho}, \hat{u})\) is defined on \([0,1]\). Then, we remark that \(\Phi(0) = u_0 - u_1\) and that \(\Phi\) is increasing. Therefore, the domain of \(\Phi\) is an interval containing 0 and there exists a unique \(b\) such that \(u(1) = u_1\). \(\square\)

Notice that a solution of (1) is constant if and only if (2) satisfies

\[ u_0 = u_1. \]

In the following, we study among other things the stability of almost constant steady solutions of (1), by which we mean solutions satisfying

\[ \exists \varepsilon > 0, \ \varepsilon \ll 1 \text{ and } |u_0 - u_1| \leq \varepsilon. \]  

(5)

For this, the following lemma will be useful.
Lemma 2.2. The unique steady solution \((\hat{\rho}, \hat{u})\) of problem (1)-(2) satisfies

\[ \hat{\rho} > 0, \hat{u} > 0, (u_1 - u_0) \hat{\rho}_x < 0, (u_1 - u_0) \hat{u}_x > 0, \]

\[ |\hat{\rho}_x|_\infty + |\hat{u}_x|_\infty \leq C(\max(u_0, u_1)) |u_0 - u_1|. \]

We denote solutions as compressive when \(\hat{u}_x > 0\) and expansive when \(\hat{u}_x < 0\).

2.2 Numerical simulations

A steady solution \((\hat{\rho}, \hat{u})\) of (1),(2) is the unique solution of system (4) where \(b\) is the unique zero of \(\Phi\). The numerical computation of such a solution is carried out in two steps:

- We compute \(b\) with a Newton’s method. We initiate the process with \(b = b_0\) where

\[ b_0 = \rho_0 u_0^2 + P(\rho_0). \]

Note that for a small viscosity \((\nu \leq 1)\), the initial starting point \(b = b_0\) ceases to be relevant. Thus, in this case, we use a dichotomy method to find a better starting point for \(b\).

- The solution of system (4) is computed with a four-order Runge-Kutta method.

We display the results of numerical simulations for a monatomic pressure law \(P(\rho) = \rho^{1.4}\) with \(\nu = 1\). Figure 1 represents the expansive solution for \(u_0 = 2, u_1 = 3\) and \(\rho_0 = 3\). Figure 2 represents the compressive solution when \(u_0 = 1.5, u_1 = 1\) and \(\rho_0 = 2\).
3 Linear estimates

3.1 The eigenvalue problem

In order to study the stability of steady states, we linearize system (1) around the steady state \((\hat{\rho}, \hat{u})\). Then, we study the corresponding eigenvalue problem for \((r, v)\):

\[
\begin{aligned}
\lambda r + (\hat{\rho}v + \hat{u}r)_x &= 0, \\
\lambda \hat{\rho}v + (\hat{\rho}uv + \hat{P}'(\hat{\rho})r)_x + \hat{u}_x (\hat{u}r + \hat{\rho}v) &= \nu v_{xx},
\end{aligned}
\]

with

\[
r(0) = v(0) = v(1) = 0.
\]

We also define the linear operator

\[
\mathcal{L}(r, v) = \begin{pmatrix}
- (\hat{\rho}v + \hat{u}r)_x \\
\nu v_{xx} - (\hat{\rho}uv + \hat{P}'(\hat{\rho})r)_x - \hat{u}_x (\hat{u}r + \hat{\rho}v)
\end{pmatrix},
\]

where \((r, v)\) satisfies (7).

Remark 3.1. For constant steady states, the eigenvalue problem simplifies into

\[
\begin{aligned}
\lambda r + \hat{\rho}v_x + \hat{u}r_x &= 0, \\
\lambda \hat{\rho}v + \hat{\rho}uv + \hat{P}'(\hat{\rho})r_x &= \nu v_{xx},
\end{aligned}
\]

In the following, we denote by \(\sigma(\mathcal{L})\) the spectrum of \(\mathcal{L}\) in \(L^2(0, 1)\). The following proposition shows that \(\sigma(\mathcal{L})\) only contains eigenvalues.

Proposition 3.2. The inverse of \(\mathcal{L}\) is compact and 0 is not an element of \(\sigma(\mathcal{L})\). Then, the set \(\sigma(\mathcal{L})\) only contains the eigenvalues of \(\mathcal{L}\).
Proof. First, we show that 0 is not in \( \sigma(\mathcal{L}) \). For \( f \) and \( g \) in \( L^2(0, 1) \), we solve

\[
\mathcal{L}(\rho, v) = (f, g)
\]

with

\[
r(0) = v(0) = v(1) = 0.
\]

This leads to the system:

\[
\begin{aligned}
&\dot{\rho} v + \dot{u}r = -\int_0^x f(y), \\
&\nu_x = \nu v_x(0) + \dot{\rho}\nu v - \frac{P'(\dot{\rho})}{u}(\dot{\rho} v + \int_0^x f) + \int_0^x g(y) - \int_0^x \dot{u} x \int_0^y f(y).
\end{aligned}
\]

Since \( v(0) = v(1) = 0 \), we can solve the second equation and we get a unique solution. Then, \( \mathcal{L} \) is invertible. Furthermore, if \( f \) and \( g \) are bounded in \( L^2(0, 1) \), we get from the previous system and the boundary conditions that \( (\hat{\rho} v + \hat{u}r) x \), \( v_x(0) \) and \( \left( v \exp \left( -\frac{1}{\nu} \int_0^x \hat{\rho} \hat{u} + \frac{\nu}{\hat{\rho}} P'(\hat{\rho}) \right) \right) x \) are bounded. Then, the result follows easily.

In order to prove the spectral stability of steady solutions, we need a high frequency estimates for problem (6)-(7). First, we establish a useful lemma.

**Lemma 3.3.** For any \( (r, v) \) satisfying the boundary conditions (7), and \(|\lambda|\) large enough, we have

\[
\begin{aligned}
|\tilde{r}|^2 &\leq \frac{C}{|\lambda|} \left( |r|^2_2 + |v|^2_2 + |(\lambda - \mathcal{L})(r, v)|^2_1 \right), \\
|r|^2_2 + |v|^2_2 &\leq \frac{C}{|\lambda|} \left( |r_x|^2_2 + |v_x|^2_2 + |(\lambda - \mathcal{L})(r, v)|_2^2 \right), \\
|v_x|^2_2 &\leq \frac{C}{|\lambda|} \left( |r_x|^2_2 + |v_x|^2_2 + |(\lambda - \mathcal{L})(r, v)|_{H^1}^2 \right),
\end{aligned}
\]

where \( \tilde{r}(x) = \int_0^x r(y)dy \).

**Proof.** If we denote \( (\lambda - \mathcal{L})(r, v) = (f, g) \) and \( \tilde{f}(x) = \int_0^x f(y)dy \), we have

\[
\begin{aligned}
\lambda \tilde{r} + \dot{\rho} v + \dot{u}r &= \tilde{f}, \\
\lambda r + (\dot{\rho} v + \dot{u}r)_x &= f, \\
\lambda \dot{\rho} v + (\dot{\rho} \nu v + P'(\dot{\rho})r)_x + \dot{u} x (\dot{u} v + \dot{\rho} v) &= \nu v_{xx} + g.
\end{aligned}
\]

Thus, we easily see that

\[
|\tilde{r}|^2_2 = \frac{1}{|\lambda|} \left| \int_0^1 \tilde{r} \lambda \tilde{r} \right| \leq \frac{C}{|\lambda|} \left( |r|^2_2 + |v|^2_2 + |\tilde{r}|^2_2 + |f|^2_2 \right).
\]

Furthermore, using the previous system and integrating by parts, we get
where for any (\rho,v) we denote (\rho,v)^\lambda=x_1(\rho,v)\lambda(x_1(\rho,v))\xi_1. This proof is based on an appropriate Goodman-type energy estimate. In the proof.

After some computations we get

\[ |r^2 + |\sqrt{\rho}v|^2 = \frac{1}{|\lambda|} \left| \int_0^1 \tau \lambda r \right| + \frac{1}{|\lambda|} \left| \int_0^1 \tau \lambda \rho v \right| \leq \frac{C}{|\lambda|} \left( |r|^2 + \left| \int_0^1 \tau \rho v \right| + |v|^2_{H^1} + |(f,g)|^2_{L^2} \right). \]

Moreover, by differentiating the second equation of (10), we obtain

\[ |(\hat{\rho}v)^2 = \frac{1}{|\lambda|} \left| \int_0^1 (\hat{\rho}v)_x \lambda(\hat{\rho}v)_x \right| \leq \frac{C}{|\lambda|} \left( |r|^2 + |v|^2_{H^2} + |\rho|_{H^1} + |v|_{H^2} + |(f,g)|^2_{L^2} \right). \]

Then, we notice that

\[
\begin{align*}
\nu v_{xx}(0) - P'(\rho_0) r_x(0) &= \rho_0 u_0 v_x(0) - g(0), \\
\nu v_{xx}(1) - P'(\rho(1)) r_x(1) &= \frac{1}{2} (\hat{\rho} x_1(\hat{\rho} v)_x^2 + \hat{\rho} v x_1(\hat{\rho} v)_x^2).
\end{align*}
\]

and thanks to Lemma A.1, we can control the boundary terms.

We can now establish a high frequency estimate in \(H^1\).

**Proposition 3.4.** Assume that \(P\) satisfies (3). There exists a constant \(\alpha > 0\) such that if \(\Re(\lambda) > -\alpha\) and \(|\lambda| > \frac{1}{\alpha}\),

\[ |(\rho,v)|^2_{H^1} \leq C |(\lambda - \mathcal{L})(\rho,v)|^2_{H^1 \times L^2}, \]

for any \((\rho,v)\) satisfying the boundary conditions (7).

**Proof.** This proof is based on an appropriate Goodman-type energy estimate. In the following we denote \((\lambda - \mathcal{L})(\rho,v) = (f,g)\). We define the energy

\[ \mathcal{E}(r,v) = \frac{1}{2} \int_0^1 \phi_1 |r_x|^2 + \phi_2 |(\hat{\rho}v)_x|^2 \]

where \(\phi_1\) and \(\phi_2\) satisfy

\[ \Phi_1 > 0, \Phi_2 > 0, \Phi_1 = P'(\hat{\rho}) \Phi_2, \frac{1}{2}(\hat{\rho} x_1(\hat{\rho} v)_x^2 + \hat{\rho} v x_1(\hat{\rho} v)_x^2) \]

This energy is equivalent to the \(H^1\)-norm by the Poincaré inequality. Then, we compute

\[ 2\Re(\lambda) \mathcal{E}(r,v) = \Re \left( \int_0^1 \phi_1 \tau \tau x_1^2 \right) + \Re \left( \int_0^1 \phi_2 \tau \tau \tau x_1^2 \right). \]

After some computations we get

\[ 2\Re(\lambda) \mathcal{E}(r,v) \leq -\nu \int_0^1 \phi_2 \nu v_{xx}^2 + \int_0^1 \left( \frac{1}{2}(\hat{\rho} x_1(\hat{\rho} v)_x^2 + \hat{\rho} v x_1(\hat{\rho} v)_x^2) \right) |r_x|^2 \]

\[ + \int_0^1 \Re(\tau v_{xx}) \left( -\hat{\rho} x_1 + \hat{\rho} P'(\hat{\rho}) \Phi_2 \right) + |f|_{H^1} |r_x|_{L^2} + |g|_{H^1} |v_{xx}|_{L^2} \]

\[ + \Re \left( \left[ -\frac{1}{2} \hat{\rho} x_1(\hat{\rho} v)_x^2 + \phi_2 \hat{\rho} x_1(\hat{\rho} v)_x^2 \right] \right) \int_0^1 \left( \hat{\rho} v x_1(\hat{\rho} v)_x^2 + \phi_2 \hat{\rho} x_1(\hat{\rho} v)_x^2 \right) \]

\[ + (|r_x|_{L^2} + |v|_{H^1})(|v|_{H^1} + |r_x|_{L^2} + |v_{xx}|_{L^2})C. \]
Then, we notice that
\[ u_0 r_x(0) = -\rho_0 v_x(0) + f(0), \]
\[ \nu v_{xx}(0) - P'(\rho_0) r_x(0) + g(0) = \rho_0 u_0 v_x(0), \]
\[ \nu v_x(1) - P'(\hat{\rho}(1)) r_x(1) + g(1) = \hat{\rho}(1) u_1 v_x(1) + P''(\hat{\rho}(1)) \hat{\rho}_x(1)r(1) + \hat{u}_x(1) \hat{u}(1)r(1). \]

(11)

Using Lemma A.1 we can control all the boundary terms except \(-\frac{1}{2} u_1 \Phi_1(1) |r_x(1)|^2\). Since \(\Phi_1 > 0\), this term has a good sign and we can get rid of it. Then, using the second and the third inequality of Lemma 3.3, we can find a constant \(\alpha > 0\), such that for \(|\lambda|\) large enough,
\[ 2 \Re(\lambda) E(r, v) \leq -\alpha |(r_x, v_{xx})|^2 + C |(\lambda - \mathcal{L})(\rho, v)|_{H^1_L}^2. \]

Since \(r(0) = v(0) = 0\), the Poincaré inequality shows that \(E\) is a norm equivalent to the \(H^1\)-norm and the inequality follows from the Poincaré-Wirtinger inequality on \(v_x\).

3.2 The linear time evolution problem

In this part, we study the linearization of system (1) around the steady state \((\hat{\rho}, \hat{u})\)
\[ \begin{cases} \partial_t (r, v) - \mathcal{L}(r, v) = 0, \\ (r, v)|_{t=0} = (r_0, v_0), \\ r(0) = v(0) = v(1) = 0. \end{cases} \]

We denote by \(e^{t\mathcal{L}}(r_0, v_0)\) the unique solution of the previous system. The following proposition shows that \(\mathcal{L}\) generates a \(C^0\)-semigroup.

Lemma 3.5. Let \(\mathcal{H} = \{(r, v) \in H^1 \times H^1, r(0) = v(0) = v(1) = 0\}\). The operator \(\mathcal{L}\), \(\text{Dom}(\mathcal{L}) = \mathcal{H} \cap (H^1 \times H^2)\), is closed densely defined on the Hilbert space \(L^2\) and generates a \(C^0\)-semigroup \(e^{t\mathcal{L}}\) satisfying \(\|e^{t\mathcal{L}}\|_{\mathcal{H}} \leq e^{\omega t}\) for some real \(\omega\). Similarly, we have \(\|e^{t\mathcal{L}}\|_{H^1_L} \leq e^{\omega t}\).

Proof. The proof is similar to one of Proposition 2.2 in [MZ03]. First we know from Proposition 3.2 that the spectrum of \(\mathcal{L}\) only contains eigenvalues. Then, for \(\lambda > 0\), we have
\[ \left| (r, \sqrt{\rho}v) \right|^2_{L^2} = \frac{1}{\lambda} \left( \lambda(r, \hat{\rho}v) (r, \hat{\rho}v) \right)_{L^2} \leq \frac{1}{\lambda} \left( C |(r, v)|_{H^1_L}^2 - \varepsilon |v_x|_{H^1}^2 + |(r, v)|_{H^1} |(\lambda - \mathcal{L})(r, v)|_{H^1} \right) \]
where we integrate by parts, we notice a good sign for \(|r(1)|^2\) and where \(\varepsilon\) is small enough.
Similarly, thanks to Lemma A.1 and the equalities (11) in the proof of Proposition 3.4, we get
\[ \left| (r_x, (\sqrt{\rho}v)_x) \right|^2_{L^2} \leq \frac{1}{\lambda} \left( C |(r, v)|_{H^1}^2 - \varepsilon |v_{xx}|_{H^1}^2 + |(r, v)|_{H^1} |(\lambda - \mathcal{L})(r, v)|_{H^1} \right) . \]
Thus, we can find a constant $\omega$ such that for any $\lambda > \omega$, we have
\[
|r(v)|_2 \leq \frac{1}{\lambda - \omega} |(\lambda - \mathcal{L})(r(v))|_2 \quad \text{and} \quad |r(v)|_{H^1} \leq \frac{1}{\lambda - \omega} |(\lambda - \mathcal{L})(r(v))|_{H^1}.
\]

The previous inequality gives a resolvent bound for $\lambda > \omega$ and it shows that $\mathcal{L}$ generates a $C^0$-semigroup by the Hille-Yosida theorem (see also [Paz83]).

The following proposition gives us a linear damping estimate under the assumption of a spectral gap.

**Proposition 3.6.** Assume that $P$ satisfies (3). Assume that there exists a constant $\alpha > 0$, such that $\Re \sigma(\mathcal{L}) < -\alpha$. Then, for any $0 < \theta < \alpha$, there exists a constant $C = C \left( \frac{1}{\alpha - \theta} \right)$, such that for any $r(0) = 0$, we have
\[
|e^{t \mathcal{L}}(r,v)|_{H^1 \times L^2} \leq Ce^{-\theta t} |(r,v)|_{H^1 \times L^2}.
\]

**Proof.** Since $\Re \sigma(\mathcal{L}) < -\alpha$, Proposition 3.4 imply that $(\lambda - \mathcal{L})^{-1}$ is uniformly bounded in $H = \{(r,v) \in H^1 \times L^2, r(0) = 0\}$ on $\mathcal{R}(\lambda) = -\alpha$. The result follows from the Prüss' theorem (see for instance [Prü84, Yos78]).

4 Spectral stability

4.1 Constant and almost constant states

First, we study the spectral stability of constant states.

**Proposition 4.1.** Assume that $(\hat{\rho}, \hat{u})$ is a constant solution of (1) and that $P$ satisfies (3). Then, there exists $\alpha > 0$, for any eigenvalue $\lambda$ of (9) with (7), $\Re(\lambda) \leq -\alpha$.

**Proof.** We compute $\Re \left( ((9)_1, P'(\hat{\rho})r)_{L^2} + ((9)_2, \hat{\rho}v)_{L^2} \right)$ and we get the following estimate
\[
\Re(\lambda) \left( \sqrt{P'(\hat{\rho})r}^2 + \sqrt{\hat{\rho}v}^2 \right) + \nu \frac{|v_x|^2}{2} + \frac{1}{2} P'(\hat{\rho}(1))u_1 |r(1)|^2 = 0.
\]

Thus, $\Re(\lambda) < 0$. Since, the spectrum of $\mathcal{L}$ in $L^2$ is also the spectrum in $L^2 \times H^1$, the result follows by Propositions 3.2 and 3.4.

We can now establish the main proposition of this part. We introduce the Evans function associated to the steady solution $(\hat{\rho}, \hat{v})$
\[
\mathcal{D}[\rho_0, u_0, u_1](\lambda) = v(1),
\]
where $(\rho, v)$ satisfies the ordinary differential equation
\[
\begin{aligned}
\rho_x &= -\frac{\lambda \rho + (\hat{\rho}v)}{\hat{u}} x + \hat{u}_x r, \\
\nu v_{xx} &= \lambda \hat{\rho} v + (\hat{\rho} w)_x + P'(\hat{\rho}) \hat{\rho}_x r - P'(\hat{\rho}) \frac{\lambda \rho + (\hat{\rho} v)}{\hat{u}} x + \hat{u}_x (\hat{u} r + \hat{\rho} v).
\end{aligned}
\]
with the initial condition
\[ r(0) = v(0) = 0 \] , \[ v'(0) = 1 \],
and \((\hat{\rho}, \hat{u})\) is a steady solution of (1) with (2). We recall that \( D[\rho_0, u_0, u_1](\lambda) = 0 \) if and only if \( \lambda \) is an eigenvalue of (6). We can now prove the spectral stability of almost constant steady solutions of (1).

**Proposition 4.2.** Assume that \( \varepsilon \) in Condition (5) is small enough and that \( P \) satisfies (3). Assume that \((\hat{\rho}, \hat{u})\) is the unique steady solution of (1) with (2). Then, an eigenvalue \( \lambda \) of (6) with (7) has a negative real part.

**Proof.** By Proposition 3.4, problem (6)-(7) does not have any eigenvalue of positive real part outside a compact set \( K \). Furthermore, from Proposition 4.1, \( D[\rho_0, u_0, u_0] \) does not have any zero inside \( K \cap \{ \Re > 0 \} \). Since the Evans function \( D \) depends continuously on the boundary conditions, \( D[\rho_0, u_0, u_1] \) never vanishes inside \( K \cap \{ \Re > 0 \} \) for \( \varepsilon \) small enough.

### 4.2 About general steady states

In the previous part, we only prove the spectral stability of almost constant states. In this part, we show some theoretical and numerical arguments that support the spectral stability of any steady states.

We know from previous works that the stability index is a necessary condition for the spectral stability (see for instance [AGJ90, PW94, GZ98]). The stability index is defined by
\[ \text{sgn} (D[\rho_0, u_0, u_1](0)) \text{sgn} (D[\rho_0, u_0, u_1](+\infty)) = 1. \]
The following proposition shows that this criterion is satisfied.

**Proposition 4.3.** The stability index is satisfied.

**Proof.** First we compute \( \text{sgn} (D[\rho_0, u_0, u_1](0)) \). After some calculations, we have the following system
\[
\begin{cases}
\hat{\rho}v + \hat{u}r = 0, \\
\hat{\rho}uv + \frac{\hat{\rho}}{\hat{u}}P'(\hat{\rho})v = \nu v_x - \nu v_x(0), \\
r(0) = v(0) = 0 \quad v_x(0) = 1,
\end{cases}
\]
and we see that \( \text{sgn} v(1) = \text{sgn} v_x(0) = 1 \). Secondly, we compute \( D[\rho_0, u_0, u_1](+\infty) \). We have the system:
\[
\begin{cases}
\lambda r + \hat{u}r_x = f, \\
\nu v_{xx} = \lambda \hat{\rho}v + P'(\hat{\rho})r_x + g, \\
r(0) = v(0) = 0 \quad v_x(0) = 1,
\end{cases} \tag{13}
\]
where \(|f|_\infty + |g|_\infty \leq C (|r|_\infty + |v|_\infty + |v_x|_\infty) \). Then, by solving the first equation of system (13) we get for \( \lambda \) large enough
\[ |r|_\infty \leq \frac{C}{\lambda} (|v|_\infty + |v_x|_\infty). \]
Then we can rewrite the second equation of system (13) as
\[ \nu v_{xx} = \lambda \hat{\rho} v + \tilde{g} \text{ with } v(0) = 0, \quad v_x(0) = 1, \]
where \( |\tilde{g}|_\infty \leq C (|v|_\infty + |v_x|_\infty) \). Then, since \( \hat{\rho} > 0 \), it easily follows that \( \text{sgn} v(1) = \text{sgn} v_x(0) = 1 \) when \( \lambda \) is large enough.

This proposition also shows that Problem (6)-(7) has an even number of nonstable eigenvalues (eigenvalues with a nonnegative real part).

Thanks to Lemma 3.4, we can numerically check that \( \sigma(L) \) does not contain nonstable eigenvalues. Such verifications have for instance been done on the whole line (see [BHRZ08]).

In the following, we display some numerical simulations for a monatomic pressure law \( P(\rho) = \rho^{4.4} \). For any \( \lambda \), we can compute the associated Evans function thanks to system (12). We use a Runge Kutta 4 scheme. For each value of \( u_0, u_1, \rho_0 \) and \( \nu \), we compute the Evans function along semi-circular contours of radius \( M \) (see Figure 3). We choose \( M \) large enough such that our domain contains the half ball of Lemma 3.3.
Figure 5: Image of a contour mapped by an Evans function. \( \nu = 0.1, u_0 = \frac{3}{2}, u_1 = 1, \rho_0 = 2 \).

Figure 4 represents the image of the contour with \( M = 10, \nu = 1, u_0 = \frac{3}{2}, u_1 = 1 \) and \( \rho_0 = 2 \). Figure 5 represents the image of the contour with \( M = 10, \nu = 0.1, u_0 = \frac{3}{2}, u_1 = 1 \) and \( \rho_0 = 2 \). We can see on these examples that the winding number of these graphs are both zero. Several computations have been performed for other values of the parameters \( \nu \in [0.1, 10], u_0 \in [1, 10], u_1 \in [1, 10] \) and \( \rho_0 \in [1, 10] \). We could not find any nonstable eigenvalues.

5 Local existence

In this section, we state a local wellposedness result for problem (1)-(2) (see, e.g., [MN82]).

Proposition 5.1. Assume that \( s \geq 1 \). Assume that \( P \) satisfies (3). Then, for any \( (\rho_{ini}, u_{ini}) \in H^s \), satisfying the boundary conditions (2) and with \( \rho_{ini} > 0 \), there exists a time \( T > 0 \), such that problem (1)-(2) has a unique solution \( (\rho, u) \) on \([0, T]\), with

\[
\sup_{[0,T]} |(\rho, u)(t)|_{H^s} \leq 2 |(\rho_{ini}, u_{ini})|_{H^s} \quad \text{and} \quad \rho(t, x) \geq \frac{\rho_{ini}(x)}{2}, \quad 0 \leq t \leq T, \quad 0 \leq x \leq 1.
\]

6 Nonlinear stability

For solution \( (\rho, u) \) of problem (1)-(2), we define \( (r, v) = (\rho - \hat{\rho}, u - \hat{u}) \). We notice that \( (r, v) \) satisfies (we remind that \( \mathcal{L} \) is defined in (8))

\[
\begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} r \\ v \end{pmatrix}_t - \mathcal{L} \begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} - (rv)_x - \hat{\rho} vv_x - rvv_x - (P(\rho) - P(\hat{\rho}) - P'(\hat{\rho}) \rho)_x \\ - (\hat{uv})_x + \hat{\rho} vv_x - \hat{uv} + (P(\rho) - P(\hat{\rho}) - P'(\hat{\rho}) \rho) \end{pmatrix}. \tag{14}
\]
Then, we get

\[
\begin{pmatrix}
1 & 0 \\
0 & \hat{\rho}
\end{pmatrix}
\begin{pmatrix}
r \\
v
\end{pmatrix}
_t - \mathcal{L}
\begin{pmatrix}
r \\
v
\end{pmatrix} = \mathcal{N},
\]

where \( \mathcal{N}_1 = -(rv)_x \) and

\[
\mathcal{N}_2 = -\frac{\hat{\rho}}{\hat{\rho} + r} \left( (\hat{u}v)_x r + \hat{\rho}vv_x + vr v_x + \left( P(r + \hat{\rho}) - P(\hat{\rho}) - P'(\hat{\rho})r \right)_x \right)
+ \frac{r}{\hat{\rho} + r} \left( (\hat{\rho}uv + P'(\hat{\rho})r)_x + \hat{u}_x (\hat{u}r + \hat{\rho}v) - \nu vv_{xx} \right).
\]

Notice that \( \mathcal{N}_1(t,0) = \mathcal{N}_2(t,0) = 0 \) and that

\[
\mathcal{N}_2(t,1) = -(uv_x r + (P'(\hat{\rho} + r) - P'(\hat{\rho}) - P''(\hat{\rho}) r)(\hat{\rho} + r)_x + P''(\hat{\rho}) r r_x) (t,1).
\]

The following proposition is a nonlinear damping estimate.

**Proposition 6.1.** Let \( T > 0 \) and consider a solution \((r,v)\) of (15) on \([0,T]\). Assume that \( P \) satisfies (3) and that there exists \( \varepsilon > 0 \) small enough such that

\[
\sup_{[0,T]} |(r,v)(t)|_{H^1} \leq \varepsilon.
\]

Then, there exists some constants \( C > 0 \) and \( \theta_0 > 0 \) such that for all \( 0 \leq t \leq T \) and any \( \theta \leq \theta_0 \),

\[
|\mathcal{N}_1(t,0)|_{H^1} \leq C e^{-\theta t} (|r,v(0)|_{H^1} + C \int_0^t e^{-\theta(t-s)} |(r,v)(s)|^2 ds).
\]

Furthermore, if

\[
\sup_{[0,T]} |(r,v)(t)|_{H^2 \times H^3} \leq \varepsilon,
\]

for \( \varepsilon \) small enough, we also have for any \( \theta \leq \theta_0 \),

\[
|(r,v)(t)|_{H^2 \times H^3}^2 \leq C e^{-\theta t} |(r,v)(0)|_{H^2 \times H^3}^2 + C \int_0^t e^{-\theta(t-s)} |(r,v)(s)|^2 ds.
\]

**Proof.** This proof is based on an appropriate Goodman-type energy estimate and is similar to the one in Proposition 3.4. We define the energy equivalent to the \( H^1 \)-norm (by the Poincaré inequality)

\[
\mathcal{E}(r,v) = \frac{1}{2} \int_0^1 \phi_1 |r_x|^2 + \phi_2 |(\hat{\rho}v)_x|^2
\]

where \( \phi_1 \) and \( \phi_2 \) satisfy

\[
\Phi_1 > 0 , \Phi_2 > 0 , \Phi_1 = P'(\hat{\rho})\Phi_2 , \frac{1}{2}(\hat{u}_x \Phi_1)_x - 2\hat{u}_x \Phi_1 < 0.
\]
Then, after some computations, we obtain

\[
\frac{d}{dt} \mathcal{E}(r, v) \leq -\nu \int_0^1 \dot{\Phi}_2 |v_{xx}|^2 + \int_0^1 \left( \frac{1}{2} (\dot{\Phi}_1)_x - 2 \dot{\Phi}_1 \right) |r_x|^2 \\
+ \int_0^1 r_x v_{xx} \left( -\dot{\Phi}_1 + \dot{\Phi} P'(\dot{\varphi}) \right) \\
+ \left[ -\frac{1}{2} \dot{\Phi}_1 |r_x|^2 + \Phi_2 \dot{\varphi} v_x \left( \nu v_{xx} - P'(\dot{\varphi}) r_x \right) \right]_0^1 \\
+ (|r|_2 + |v|_{H^1}) (|v|_{H^1} + |r_x|_2 + |v_{xx}|_2) C \\
+ \int_0^1 \phi_1 r_x (\mathcal{N}_1)_x + \phi_2 (\dot{\varphi} v)_x (\mathcal{N}_2)_x.
\]

Integrating by parts, using Lemma A.1 and the fact that \( |(r, v)|_{H^1} \) is small enough we get

\[
\int_0^1 \phi_1 r_x (\mathcal{N}_1)_x + \phi_2 (\dot{\varphi} v)_x (\mathcal{N}_2)_x \leq [\Phi_2 \dot{\varphi} v_{xx}]_0^1 + C \ |(r, v)|_{H^1}^2 (|(r, v)|_{H^1} + |v_{xx}|_2).
\]

Then, since \( \mathcal{N}_1(t, 0) = \mathcal{N}_2(t, 0) = 0 \), we have

\[
u_0_0 r_x(t, 0) = -\rho_0 v_x(t, 0), \\
\nu_0 v_{xx}(t, 0) - P'(\rho_0) r_x(t, 0) = \rho_0 u_0 v_x(t, 0), \\
\nu_0 v_{xx}(t, 1) - P'(\rho(1)) r_x(t, 1) + \mathcal{N}_2(t, 1) = \dot{\rho}(1) u_1 v_x(t, 1) + P''(\rho(1)) \dot{\varphi}_x(t, 1) r(t, 1) + \dot{\varphi}_x(t, 1) u_1 r(t, 1).
\]

Finally, thanks to the previous boundary equalities, Lemma A.1, Lemma A.2 and the fact that \( |(r, v)|_{H^1} \) is small enough, we obtain

\[
\frac{d}{dt} \mathcal{E}(r, v) \leq -\theta_0 |(r_x, v_x)|_2^2 + C |(r, v)|_2^2.
\]

Then, the first inequality easily follows. Similarly, by considering \( (r_t, v_t) \), we get

\[
\frac{d}{dt} (\mathcal{E}(r, v) + \mathcal{E}(r_t, v_t)) \leq -\theta_0 \left( |(r_x, v_x)|_2^2 + |(r_t, v_t)|_2^2 \right) + C |(r, v)|_2^2.
\]

(16)

Then, using (14), we can see that for \( \varepsilon \) small enough,

\[
|(r, v)|_{H^2 \times H^3} \leq ((|(r, v)|_{H^1} + |(r_t, v_t)|_{H^1}) C,
\]

and the second inequality follows easily.

\[\square\]

**Remark 6.2.** By taking further time-derivatives, we could obtain an estimate similar to (16) in an arbitrarily high-regularity Sobolev space of mixed type \( H^r \times H^s \), with \( s \sim 2r \) as \( r \to \infty \). This observation repairs a minor error in [CHNZ09], citing an estimate with \( r = s \).

We can now state the main result of this paper.
Theorem 6.3. Assume that $P$ satisfies (3). Assume that there exists a constant $\alpha > 0$ such that $\Re(\sigma(L)) < -\alpha$. Then, there exists $\varepsilon > 0$ and $\theta > 0$, for any $(\rho_{ini}, u_{ini}) \in H^2 \times H^3$ satisfying the boundary conditions (2) and

$$|(\rho_{ini}, u_{ini}) - (\hat{\rho}, \hat{u})|_{H^2 \times H^3} \leq \varepsilon,$$

the unique solution $(\rho, u)$ of problem (1)-(2) with the initial condition $(\rho_{ini}, u_{ini})$ satisfies

$$|(\rho(t), u(t)) - (\hat{\rho}, \hat{u})|_{H^2 \times H^3} \leq C |(\rho_{ini}, u_{ini}) - (\hat{\rho}, \hat{u})|_{H^2 \times H^3} e^{-\theta t}.$$

Remark 6.4. As we will see in the proof, since we do not know if $N_2(t, 1) = 0$, the only way to use a linear damping estimate is to work in $L^2$ for the $v$ component. That is why in Proposition 3.6, we used $H^1 \times L^2$ and not $H^1$.

Proof. We denote by $U(t, x) = (\rho(t), u(t)) - (\hat{\rho}, \hat{u})(t, x)$. In the following, We assume that $\varepsilon$ is small enough. $U$ exists on $[0, T]$ by Proposition 5.1. The Duhamel formulation of Equation (15) is

$$U(t) = e^t L U(0) + \int_0^t e^{(t-s) L} \mathcal{N}(s) ds.$$

Noticing that $\mathcal{N} \in \{(r, v) \in H^1 \times L^2, r(0) = 0\}$, we can use Proposition 3.6 together Proposition 5.1, and there exists $\theta > 0$,

$$|U(t)|_2 \leq |U(t)|_{H^1 \times L^2} \leq C e^{-\theta t} |U(0)|_{H^1 \times L^2} + \int_0^t C |(U(s)|_{H^2}) e^{-\theta(t-s)} \left( |U(s)|_{H^2}^2 \right) ds.$$

Then, Equality (14) together with Proposition 5.1 gives us

$$|U(t)|_2 \leq C e^{-\theta t} |U(0)|_{H^1 \times L^2} + \int_0^t C |(U(s)|_{H^2}) e^{-\theta(t-s)} \left( |U(s)|_{H^1}^2 + |U_t(s)|_{H^1}^2 \right) ds.$$

Proposition 5.1 allows us to use the nonlinear damping estimate in Proposition 6.1 and we get for $\theta > 0$ small enough,

$$|U(t)|_2 \leq C \left( |U(0)|_{H^2} \right) \left( 1 + t e^{-\theta t} |U(0)|_{H^2 \times H^3} + \int_0^t (t-s) e^{-\theta(t-s)} |U(s)|_{H^2}^2 ds \right).$$

Denoting $\zeta_0(t) = \sup_{[0,T]} e^{\frac{\theta t}{2}} |U(t)|_2$, we obtain that for $0 \leq t \leq T$,

$$\zeta_0(t) \leq C \left( |U(0)|_{H^2} \right) (\varepsilon + \zeta_0(t)^2).$$

Furthermore, denoting $\zeta_1(t) = \sup_{[0,T]} \left( e^{\frac{\theta t}{2}} |U(t)|_{H^1} + e^{\frac{\theta t}{2}} |U_t(t)|_{H^1} \right)$ and using Proposition 6.1, we see that $\zeta_1$ is bounded on $[0, T]$. Finally, if $\varepsilon$ is small enough, we can take $T = +\infty$ and $\zeta_1$ is bounded on $\mathbb{R}^+$. \qed
7 An improvement in some situations

The main result of this paper, Theorem 6.3, states that spectrally stable steady states are stable in $H^2 \times H^3$. In this part, we prove that under more restrictive conditions, we can state a stability result in $H^1 \times H^2$. To achieve that, we add another assumption:

$$P'' > 0 \text{ if } \hat{u}_x > 0 \text{ (compressive solutions)},$$

$$\frac{P''(y)}{P'(y)} < \frac{2}{y} \text{ and } \hat{\rho}_x < \frac{1}{4} \hat{\rho} \text{ if } \hat{u}_x < 0 \text{ (small expansive solutions)}. \quad (17)$$

With this additional assumption, we can establish a high frequency estimate in $L^2$.

**Proposition 7.1.** Assume that $P$ satisfies (3) and that Condition (17) is satisfied. There exists a constant $\alpha > 0$ such that if $\Re(\lambda) > -\alpha$ and $|\lambda| > \frac{1}{\alpha}$,

$$|\langle \rho, v \rangle|_2^2 \leq C \langle \lambda - L \rangle \langle \rho, v \rangle |_2^2,$$

for any $(\rho, v)$ satisfying the boundary conditions (7).

**Proof.** This proof is based on an appropriate Goodman-type energy estimate. In the following we denote $(\lambda - L) \langle \rho, v \rangle = \langle f, g \rangle$. We define the following energy

$$E(r, v) = \frac{1}{2} \int_0^1 \phi_1 |r|^2 + \phi_2 \hat{\rho} |v|^2$$

where $\phi_1$ and $\phi_2$ satisfy

$$\Phi_1 > 0, \Phi_2 > 0, \hat{\rho} \Phi_1 = P'(\hat{\rho}) \Phi_2.$$

Then, we compute

$$2 \Re(\lambda) E(r, v) = \Re \left( \int_0^1 \phi_1 \tau \lambda r \right) + \Re \left( \int_0^1 \phi_2 \hat{\rho} \tau \lambda v \right).$$

After some computations we get

$$2 \Re(\lambda) E(r, v) \leq -\nu \int_0^1 \Phi_2 |v_x|^2 + \frac{1}{2} \int_0^1 \hat{u}^2 \left( \frac{\phi_1}{u} \right) |r|^2$$

$$+ \int_0^1 \Re(\tau v_x) (P'(\hat{\rho}) \Phi_2 - \hat{\rho} \Phi_1) + C \langle f, g \rangle |(r, v)|_2$$

$$+ \frac{1}{2} \int_0^1 (\nu(\Phi_2)_{xx} - 2 \Phi_2 \hat{u}_x \hat{\rho} + (\Phi_2)_{x} \hat{\rho} \hat{u}) |v|^2$$

$$+ \int_0^1 \Re(\tau) \left( (\Phi_2)_{x} P'(\hat{\rho}) - \Phi_1 \hat{\rho}_x - \Phi_2 \hat{u}_x \hat{u} \right).$$

Then, we separately consider the three situations $\hat{u}_x > 0$ (compressive solution), $\hat{u}_x = 0$ (constant solution) and $\hat{u}_x < 0$ (expansive solution).
and the inequality follows. Thus, using the first inequality of Lemma 3.3, we can find a constant $\alpha > 0$, on $[0, T]$ and

$$\hat{u}(\Phi_1) - \hat{u}_x \Phi_1 = \frac{P''(\rho)\hat{\rho}_x \hat{u}}{\hat{\rho}} < 0, \nu(\Phi_2)_{xx} - 2\Phi_2 \hat{u}_x \hat{\rho} + (\Phi_2)_x \hat{\rho} \hat{u} = -2\hat{u}_x \hat{\rho} < 0.$$  

- If $\hat{u}_x = 0$, we take $\phi_1 = P'(\rho) - \beta x$, $\phi_2 = \hat{\rho} - \beta \frac{\hat{\rho}}{P'(\rho)} x$ with $\beta > 0$ small enough, and we get

$$\hat{u}(\Phi_1) - \hat{u}_x \Phi_1 = -\beta \hat{u} < 0, \nu(\Phi_2)_{xx} - 2\Phi_2 \hat{u}_x \hat{\rho} + (\Phi_2)_x \hat{\rho} \hat{u} = -\beta \frac{\hat{\rho}^2 \hat{u}}{P'(\rho)} < 0.$$  

- If $\hat{u}_x < 0$, we take $\Phi_2(x) = \sqrt{M - 2x}$, $M > 2$ and $\Phi_1(x) = \frac{P'(\rho)}{\rho} \Phi_2(x)$ and thanks to Condition (17) we get

$$\hat{u}(\Phi_1) - \hat{u}_x \Phi_1 = \frac{\Phi_2}{\rho} P'(\rho) \hat{\rho} \hat{u} \left( \frac{P''(\rho)\hat{\rho}_x - \frac{1}{M - 2x}}{P'(\rho)} \right) < 0,$$

$$\nu(\Phi_2)_{xx} - 2\Phi_2 \hat{u}_x \hat{\rho} + (\Phi_2)_x \hat{\rho} \hat{u} \leq \Phi_2 \hat{u} \left( 2\hat{\rho}_x - \frac{\hat{\rho}}{M - 2x} \right) < 0.$$  

Moreover, in any case, we have

$$\int_0^1 \Re \left( v^T \left( (\Phi_2)_{xx} P'(\rho) - \Phi_1 \hat{\rho}_x - \Phi_2 \hat{u}_x \hat{\rho} \right) \right) \leq C |\hat{\rho}| |v|_{H^1}.$$  

Thus, using the first inequality of Lemma 3.3, we can find a constant $\alpha > 0$, for $|\lambda|$ large enough,

$$2\Re(\lambda)E(r, v) \leq -\alpha |(r, v)|^2 + C |(\lambda - \mathcal{L})(\rho, v)|^2,$$

and the inequality follows. 

Thanks to this $L^2$ high frequency estimate, we can improve Proposition 3.6. Under the assumption that $Re(\sigma(\mathcal{L})) \leq -\alpha < 0$, we get

$$|e^{\mathcal{L}t}(r, v)|_2 \leq Ce^{-\alpha t} |(r, v)|_2.$$  

Furthermore, thanks to the previous appropriate Goodman-type estimate, we can improve the nonlinear damping estimate in Proposition 6.1. If $(r, v)$ is a solution of (15) on $[0, T]$ and

$$\sup_{[0, T]} |(r, v)(t)|_{H^1 \times H^2} \leq \varepsilon,$$

for $\varepsilon$ small enough, we have

$$|(r_t, v_t)(t)|_2^2 \leq Ce^{-\theta t} |(r, v)(0)|^2_{H^1 \times H^2} + C \int_0^t e^{-\theta(t-s)} |(r, v)(s)|^2 ds.$$  

Finally, applying the Duhamel formulation in $L^2$, we obtain the following theorem.
Theorem 7.2. Assume that $P$ satisfies (3) and that Condition (17) is satisfied. Assume that there exists a constant $\alpha > 0$ such that $\Re(\sigma(L)) < -\alpha$. Then, there exists $\varepsilon > 0$ and $\theta > 0$, for any $(\rho_{ini}, u_{ini}) \in H^1 \times H^2$ satisfying the boundary conditions (2) and
\[
|\((\rho_{ini}, u_{ini}) - (\hat{\rho}, \hat{u})\)|_{H^1 \times H^2} \leq \varepsilon
\]
the unique solution $(\rho, u)$ of problem (1)-(2) with the initial condition $(\rho_{ini}, u_{ini})$ satisfies
\[
|\((\rho, u)(t) - (\hat{\rho}, \hat{u})\)|_{H^1 \times H^2} \leq C|\((\rho_{ini}, u_{ini}) - (\hat{\rho}, \hat{u})\)|_{H^1 \times H^2} e^{-\theta t}.
\]

A $L^\infty$ estimates and interpolation

In this appendix, we recall some basic results about Sobolev spaces in a bounded domain. The following lemma allows us to control boundary terms and $L^\infty$-norms by appropriate Sobolev norms.

Lemma A.1. For any $f \in H^1(0, 1)$, we have
\[
|f|_\infty \leq \sqrt{2}|f|_2|f'_x|, \text{ if } f(0) = 0,
\]
\[
|f|_\infty \leq |f|_2 + \sqrt{2}|f|_2|f'_x|.
\]

Proof. For any $x, y \in [0, 1]$, we have
\[
f(x)^2 = f(y)^2 + 2\int_y^x f(z)f'(z)dz.
\]
If $f(0) = 0$, we choose $y = 0$ and we obtain the result by integrating over $x$. If not, we integrate over $y$ and then over $x$ and we get the result.

To this, we add the following derivative-interpolation theorem.

Lemma A.2. For any $v \in H^2(0, 1)$,
\[
|v_x^2|_2 \leq \left(|v|^2_2 + |v|_2|v_{xx}|_2\right)C.
\]

Proof. Integrating by parts, we get
\[
\int_0^1 v_x^2dx = -\int_0^1 vv_{xx}dx + [vv_x]_0^1.
\]
The result follows from the previous Lemma and the Young’s inequality.
References


