The Topological Hochschild Homology of the Gaussian Integers

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§0. Introduction

This paper gives an explicit calculation of the 2-torsion in the topological Hochschild homology $\text{THH}$ of rings of integers in quadratic extensions of the rationals which are ramified at the prime 2 (Theorem (1.14)). The calculation of the $p$-torsion in $\text{THH}$ of rings of integers in extensions unramified at $p$ can be deduced from Bökstedt’s calculation of this invariant for the integers (see (2.2)). This means that given the results of this paper, the 2-torsion of $\text{THH}$ is known for any quadratic number ring. It also means that for rings of integers in extensions which ramify only at the prime 2, specifically, the Gaussian integers, $\mathbb{Z}[\sqrt{2}]$, and $\mathbb{Z}[\sqrt{-2}]$, the homotopy type of the $\text{THH}$ spectrum is completely known, and given in

Theorem (2.3):

$$\text{THH}(\mathbb{Z}[i]) \simeq K(\mathbb{Z}, 0)^2 \times \prod_{j=1}^{\infty} K(\mathbb{Z}/2j\mathbb{Z}, 2j-1)^2$$

$$\text{THH}(\mathbb{Z}[\sqrt{\pm 2}]) \simeq K(\mathbb{Z}, 0)^2 \times \prod_{j=1}^{\infty} (K(\mathbb{Z}/2j\mathbb{Z}, 2j-1) \times K(\mathbb{Z}/4j\mathbb{Z}, 2j-1)).$$

In comparison, Bökstedt’s calculation in [4] gives

$$\text{THH}(\mathbb{Z}) \simeq K(\mathbb{Z}, 0) \times \prod_{j=0}^{\infty} K(\mathbb{Z}/j\mathbb{Z}, 2j-1),$$

and it turns out that the standard inclusion $\mathbb{Z} \hookrightarrow R$ induces an isomorphism into one copy of $\mathbb{Z}$ on the 0-dimensional stable homotopy and multiplication by 2 from $\mathbb{Z}/j\mathbb{Z}$ into $\mathbb{Z}/2j\mathbb{Z}$ in higher dimensions. The standard inclusion of a number ring $R$ into its completion at 2 induces an isomorphism of stable homotopy groups with coefficients mod 2, $\pi_0^S(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \cong \pi_0^S(\text{THH}(\hat{R}_2); \mathbb{Z}/2\mathbb{Z})$. The spectrum $\text{THH}(\hat{R}_2)$ is interesting in light of the result of Hesselholt and Madsen in [12], which tells us that for completions of number rings at a prime $p$, the topological cyclic homology spectrum and the algebraic K-theory spectrum become the same after completion at $p$. Topological cyclic homology, introduced by Bökstedt, Hsiang, and Madsen in [5], is a suitably defined fixedpoint set of the circle action on topological Hochschild homology. And thus, the calculation of the
topological Hochschild homology of the extensions discussed in this paper is a step towards understanding the algebraic K-theory of their completions, in particular at the prime 2. Further results, by the author and Madsen ([15]), now calculate the topological Hochschild homology of arbitrary rings of integers in algebraic extensions of \( \mathbb{Q} \).

The topological Hochschild homology \( \text{THH} \) of an associative ring \( R \) with unit was introduced by Bökstedt in [3]. It is a ‘fattened’ spectrum version of Hochschild homology, which was constructed in order to give the Dennis trace map \( K_n(R) \to \text{HH}_n(R) \) a better target, from which more information about \( K_*(R) \) could be pulled back. Goodwillie conjectured the existence of \( \text{THH} \) before it was defined, on the basis of his calculus of functors, and he conjectured that the stable homotopy groups should be isomorphic to the stable K-groups of \( R \). After Bökstedt’s definition of \( \text{THH} \), Dundas and McCarthy confirmed the isomorphism of these invariants in [9]. Pirashvili and Waldhausen have further shown that these stable homotopy groups are isomorphic to the MacLane homology groups of \( R \) ([21]). Betley and Pirashvili in [2] have shown that for a fixed ring \( R \), the MacLane homology groups of \( R \), with coefficients in functors from the category of finitely generated free \( R \)-modules cross its opposite, can be thought of as derived functors of the 0’th MacLane homology group. Since regular MacLane homology is a special case of this construction, their work gives a homological-algebra definition of MacLane homology, and hence also of \( \text{THH} \). Bökstedt’s definition of \( \text{THH} \) uses a formalism of functors with smash product (FSPs); a different approach, of defining \( \text{THH} \) on a category of spectra with a strictly associative smash product, has since been defined and used ([18], [10]). Work in progress in Chicago and Osnabrück makes it clear that the two approaches are equivalent ([17]). In the case of Eilenberg-MacLane spectra of discrete rings, however, Bökstedt’s FSPs give smaller spaces, which are convenient to work with, and they shall be used in this paper. In [4] (see also Breen [7]), Bökstedt calculated \( \text{THH}(\mathbb{Z}) \) and \( \text{THH}(\mathbb{Z}/p\mathbb{Z}) \) for \( p \) prime. Both of these proved much more interesting, and in the case of \( \mathbb{Z} \) much more suggestive of torsion structure of \( K_*(\mathbb{Z}) \), than the respective Hochschild homology groups (which vanish in positive dimensions). Results by Hesselholt on group rings and results in [14] extended the range of rings for which \( \text{THH} \) is known— in fact, both are special cases of a splitting phenomenon now described by Hesselholt and Madsen in section 6.1 of [12]: when \( R \) is an \( A \)-algebra spanned over \( A \) by linearly independent elements \( x_1, \ldots x_n \) such that the set \( \{0, x_1, \ldots x_n\} \) is closed under multiplication, \( \text{THH}(R) \) is the smash product of \( \text{THH}(A) \) with a cyclic nerve complex constructed from the \( x_i \)'s (if \( R = A[G] \), this complex is homotopy equivalent to the free loopspace \( \wedge BG \)). Pirashvili in [20] makes the same calculation in terms of MacLane homology, and his calculation generalizes to smooth algebras over \( \mathbb{Z} \), where the collapse at \( E^2 \) of the spectral sequence he uses continues to hold.

It can be seen from the splitting above that in the case of group rings \( R = \mathbb{Z}[G] \), a copy of \( \text{THH}(\mathbb{Z}) \) sits inside \( \text{THH}(R) \) as a direct summand— from functoriality, it is clear that this should be the case simply because \( R \) is in this case augmented over \( \mathbb{Z} \). It was not, however, clear that in the case of number rings there would not be similar behavior. Bökstedt constructs a spectral sequence for calculating \( H^S_*(\text{THH}(R);\mathbb{Z}/p\mathbb{Z}) \) for \( R \) which is additively free over \( \mathbb{Z} \). The \( E^2 \)-term of this spectral sequence splits

\[
E^2_{*,*}(\text{THH}(R);\mathbb{Z}/p\mathbb{Z}) \cong E^2_{*,*}(\text{THH}(\mathbb{Z});\mathbb{Z}/p\mathbb{Z}) \otimes \text{HH}_*(R;\mathbb{Z}/p\mathbb{Z})
\]
and there was no known case where this splitting did not last to the $E^\infty$ term. For those rings where the splitting does last to the $E^\infty$ term, topological Hochschild homology is exactly as fine, as an invariant, as the Hochchild homology groups (taken over all choices of coefficients). This paper shows that such a splitting at the $E^\infty$ level does not occur for $p = 2$ when $R$ is the ring of integers in a quadratic extension of the rationals which ramifies at 2. In fact, in this case the spectral sequence differentials on (0.0.1) are as non-trivial as they could possibly be, given the constraint that once we reduce modulo 2 we know by [14] that

$$E^\infty_{s,*}(\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \cong E^\infty_{s,*}(\text{THH}(\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \otimes \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}).$$

The calculation of these spectral sequence differentials forms the core of the calculation of the 2-torsion in $\text{THH}(R)$ for these rings in theorem (1.14). As mentioned earlier, this concludes the calculation of the stable homotopy groups of $\text{THH}(R)$ when $R$ is ramified only at 2; other cases require further results which will appear in [15] to conclude the calculation. The stable homotopy groups in the result (2.3) mentioned above are much smaller than the analogous ones for, say, the group ring $R = \mathbb{Z}[x]/(x^2 - 1)$, though the Hochschild homology groups, with any choice of coefficients, coincide. Thus topological Hochschild homology is a fundamentally deeper invariant than Hochschild homology, not only in the base cases of $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$, but also when studying extensions.

The first section of this paper describes the calculation of the 2-torsion in topological Hochschild homology of quadratic number rings in extensions of the rationals which ramify at 2, modulo some technical lemmas which are proved in the last section. The second section uses the results of the first to work out explicitly the topological Hochschild homology of quadratic number rings which ramify only at 2. The last section describes in more detail the setup which is used for the calculations of the first section, as well as proving the technical lemmas used there.

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§1. The Homology Calculation for Quadratic Rings Which Ramify at Two

We will use Bökstedt’s setup from [4], with the indexing of section 1.7 in [12] and the notations of [14]. These will be reviewed in the beginning of section (3.1). Bökstedt constructs a spectral sequence converging to $H^S_*(\text{THH}(R))$, with

(1.0.1) \hspace{1cm} E^2_{s,*} \cong \text{HH}_*(H^S_*(\text{K}(R, 0); \mathbb{Z}/p\mathbb{Z})) \cong \text{HH}_*(H^S_*(\text{K}(\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z})) \otimes \text{HH}_*(R; \mathbb{Z}/p\mathbb{Z})

if $R$ is additively a free finitely spanned $\mathbb{Z}$-module, and

(1.0.2) \hspace{1cm} E^2_{s,*} \cong \text{HH}_*(H^S_*(\text{K}(R, 0); \mathbb{Z}/p\mathbb{Z})) \cong \text{HH}_*(H^S_*(\text{K}(\mathbb{Z}/p\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z})) \otimes \text{HH}_*(R)$
if $R$ is additively a finite dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ (all rings are assumed to have units, but are not necessarily commutative).

As explained in the introduction, the Hochschild homology factor in (1.0.1) and (1.0.2) lives to the $E^\infty$ term in the cases worked out in [12] and [14]. We will see that this is not the case if $R$ is the ring of integers in a quadratic extension of the rationals which ramifies at 2 and $p = 2$.

(1.1) Outline of the Calculation for the Ramified Case:

Let $R$ be the ring of integers in a quadratic extension of the rationals which is ramified at $p = 2$. Consider the $E^2$ term (1.0.1). By [4], the $E^2$ term for $\text{THH}(\mathbb{Z})$ is

$$\text{HH}_*(H_*^S(K(\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z})) = \mathcal{A}[e_3, e_4, e_8, \ldots]/(e_i^2), \quad \dim(e_i) = i,$$

where $\mathcal{A} = H_*^S(K(\mathbb{Z}, 0); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\eta, \bar{\xi}_2, \bar{\xi}_3, \ldots]$. By [13],

$$\text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) = \text{HH}_*(R \otimes \mathbb{Z}/2\mathbb{Z}) \cong R/2R[\epsilon]/(\epsilon^2) \otimes \Gamma(a_2), \quad \dim(\epsilon) = 1, \quad \dim(a_2) = 2,$$

where $\Gamma$ denotes the divided power algebra. Thus (1.0.1) can be rewritten as

$$E^2_{*, *}_* = \mathcal{A}[e_3, e_4, e_8, \ldots]/(e_i^2) \otimes R[\epsilon, a_2, a_2^{(2)}, a_2^{(4)}, \ldots]/(\epsilon^2, (a_2^{(i)})^2).$$

The elements of $\mathcal{A}$ have filtration degree zero, the $\bar{e}_i$ and $\epsilon$ filtration degree one, so $d_2$ and all higher spectral sequence differentials must vanish on them. It will be shown that $d_2$ must also vanish on all the generators $a_2^{(i)}$ for dimension reasons, and the higher spectral sequence differentials all vanish on $a_2$ since it is of filtration degree two. The most important step in the calculation will be to demonstrate directly that for $i \geq 2$, $d_3(a_2^{(i)}) = e_3a_2^{(i-2)}$; this shows that all monomials involving $a_2^{(i)}$, for $i \geq 2$, but not involving $e_3$, do not survive to the $E^4$ term, since they are not in $\ker d_3$, and on the other hand, neither do any terms involving $e_3$, since they are in $\text{im} d_3$. The result will be that

$$E^4_{*, *}_* \cong \mathcal{A} \otimes R[\epsilon, a_2, e_4, e_8, \ldots]/(\epsilon^2, a_2^2, e_i^2),$$

and because of the filtration degree of the generators of this term, $E^4 = E^\infty$. Since the calculation is done over a field, the $E^\infty$ term gives the correct linear structure of the stable homology, and the multiplication of the homology of the associated graded object— we will show that the actual multiplicative structure on the stable homology is

$$H_*^S(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A} \otimes R[\bar{\epsilon}, a_2]/(\bar{\epsilon}^2),$$

with $a_2^{2i-1}$ representing $e_{2i}$ in the $E^\infty$ term. Recall from [4] that $H_*^S(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A}[ar{e}_3, \bar{e}_4]/(\bar{e}_3^2)$; the inclusion $\mathbb{Z} \hookrightarrow R$ sends $\bar{e}_3 \mapsto 0, \bar{e}_4 \mapsto a_2^2$. Knowing the ring structure of stable homology makes it easier to calculate the Bockstein operators on its classes. Since $\text{THH}(R)$ for finitely generated $R$ is known a priori to be a product of Eilenberg-MacLane spectra, its homotopy can be read off its stable homology (as a $\mathcal{A}$-module, with action of the Bockstein operators).

To begin to carry out this program, we need
Lemma (1.2) If $R$ is the ring of integers in a quadratic extension of the rationals which ramifies at 2, there is an element $\zeta \in R$ such that $\zeta$ and 1 span $R$ linearly over $\mathbb{Z}$, $\zeta^2$ is divisible by 2 in $R$, and $\zeta^2/2$ is invertible in $\hat{R}_2$.

Proof. Note that to satisfy the second and third requirements of the lemma, it is enough to know that the 2-adic valuation of the $\zeta$ we will pick is 1/2 (all elements of a fixed valuation are products by $\hat{R}_2$-units of each other, and for any element of $R$ which is divisible by 2 in $\hat{R}_2$, the quotient of that element by 2 must be in $R$). So we start with any $\hat{\zeta} \in R$ with valuation 1/2; these must exist since $R$ is dense in the completion $\hat{R}_2$, and $\hat{R}_2$ is a principal ideal domain in which the ideal 2$R$, of elements whose valuation is $\geq 1$, satisfies 2$R$ = $I^2$ for some ideal $I$. The valuation of $\hat{\zeta} + 2a$ will of course be 1/2 as well, for any $a \in R$, and the claim is now that we can pick an $a$ so that $\zeta = \hat{\zeta} + 2a$ will satisfy the first condition. We know there is some $x \in R$ which, together with 1, spans $R$ linearly over $\mathbb{Z}$ and so we look at $R/2R$, which consists of the four classes 2$R$, 1 + 2$R$, x + 2$R$, and 1 + x + 2$R$. The elements of the first class have valuations $\geq 1$, those of the second class valuations of 0. Thus $\zeta$ belongs to the third or fourth class, and so there some element $a \in R$ for which $\hat{\zeta} + 2a = x$ or $\hat{\zeta} + 2a = 1 + x$.

We define

(1.2.1) \[ \iota = \zeta^2/2, \]

which exists in $R$ by our choice of $\zeta$. In the case of the Gaussian integers, we can for example think of $\zeta = 1 + i$, and then $\iota = i$. Note that by our construction, $\zeta$ satisfies a minimal polynomial over $\mathbb{Z}$ of the form $f(x) = x^2 + 2ax + 2b = 0$; in the final formula, it will become important to know the parity of $a$.

(1.3) By [4], we know that Bökstedt’s spectral sequences calculating $H^S_*(\text{THH}(\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$ and $H^S_*(\text{THH}(\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$ collapse at the $E^2$ term; in the first case we have

(1.3.1) \[ H^S_*(K(\mathbb{Z}, 0); \mathbb{Z}/2\mathbb{Z}) = \mathcal{A} = \mathbb{Z}/2\mathbb{Z}[\eta, \xi_2, \xi_3, \ldots], \quad \deg \eta = 2, \quad \deg \xi_i = 2^i - 1 \]

and so

(1.3.2) \[ E^\infty_{*,*} = E^2_{*,*} = HH_*(\mathcal{A}) \cong \mathcal{A}[e_3, e_4, e_8, \ldots]/(e_i^2) \]

where $e_3$ is represented by $1 \otimes \eta$ and $e_{2^i}$ by $1 \otimes \xi_i$. In the second case we have

\[ H^S_*(K(\mathbb{Z}/2\mathbb{Z}, 0); \mathbb{Z}/2\mathbb{Z}) = \mathcal{A} = \mathbb{Z}/2\mathbb{Z}[\xi_1, \xi_2, \xi_3, \ldots], \quad \deg \xi_i = 2^i - 1 \]

and so

\[ E^\infty_{*,*} = E^2_{*,*} = HH_*(\mathcal{A}) \cong \mathcal{A}[\lambda_1, \lambda_2, \ldots]/(\lambda_i^2) \]

where $\lambda_i$ is represented by $1 \otimes \xi_i$. For calculations later in the paper, it will be important to specify our exact choice of $\xi_i$ and $\xi_i$. We do not take $\xi_i$ to be the standard Milnor generators, but rather their images under the canonical anti-automorphism of the dual of the Steenrod algebra, which are called $\chi \xi_i$ in the notation of [19] and [8]. The reduction
map \( \mathbb{Z} \hookrightarrow \mathbb{Z}/2\mathbb{Z} \) induces an inclusion \( \mathcal{A} \hookrightarrow \mathcal{A} \), and since corollary 2.5 in chapter III of [8] gives us the image of this inclusion map explicitly, we take the \( \eta \) of (1.3.1) to be the pre-image of \( \zeta_1^2 \), and the \( \xi_i \) there (for \( i > 1 \)) to be the pre-images of the respective \( \xi_i \).

Note that the above formulae refer to a multiplicative structure on Bökstedt’s spectral sequence—this is in fact induced by a multiplicative structure

\[
(1.3.3) \quad M : \text{THH}^{(m)}(R) \times \text{THH}^{(n)}(R) \to \text{THH}^{(m+n)}(R)
\]

defined in section 1.7 of [12], which induces the shuffle product on the \( E^k \) terms for \( k \geq 1 \) (see [14]). For such \( k \), we also know that this product commutes with the maps \( E_{*,*}^k(\text{THH}^{(m)}(R); \mathbb{Z}/2\mathbb{Z}) \to E_{*,*,+\ell}^k(\text{THH}^{(m+\ell)}(R); \mathbb{Z}/2\mathbb{Z}) \) induced by the spectrum suspension maps, and so when we want to calculate a product of two stable filtered homology classes we can pick representatives for them on any \( \text{THH}^{(m)}, \text{THH}^{(n)} \) that we please.

By our choice of generators, the reduction map \( \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \) induces a map sending \( \eta \to \xi_1^2 \) and \( \xi_i \to \xi_i \). So on the \( E^2 \) terms we get a map

\[
(1.3.4) \quad \bar{\mathcal{A}}[e_3, e_4, e_8, \ldots]/(e_7^2) \to \mathcal{A}[\lambda_1, \lambda_2, \ldots]/(\lambda_i^2)
\]

with \( \bar{\mathcal{A}} \hookrightarrow \mathcal{A}, e_3 \mapsto 0, \) and \( e_2i \mapsto \lambda_i \). The standard inclusion of \( \mathbb{Z}/2\mathbb{Z} \) in \( R \otimes \mathbb{Z}/2\mathbb{Z} \) is obtained from the standard inclusion of \( \mathbb{Z} \) in \( R \) by tensoring mod 2, and thus the reduction map \( R \to R \otimes \mathbb{Z}/2\mathbb{Z} \) induces the map of (1.3.4) on the first factor of

\[
(1.3.5) \quad \text{HH}_*(H_*^S(K(R, 0); \mathbb{Z}/2\mathbb{Z})) \cong \text{HH}_*(\bar{\mathcal{A}}) \otimes \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\text{red}} \text{HH}_*(H_*^S(K(R \otimes \mathbb{Z}/2\mathbb{Z}, 0); \mathbb{Z}/2\mathbb{Z})) \cong \text{HH}_*(\mathcal{A}) \otimes \text{HH}_*(R \otimes \mathbb{Z}/2\mathbb{Z}).
\]

On the second factor, red induces the obvious isomorphism \( \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) \cong \text{HH}_*(R \otimes \mathbb{Z}/2\mathbb{Z}) \).

The right side of (1.3.5) is the \( E^2 \) term of a spectral sequence which collapses at \( E^2 \) by [14]—in particular, we know that all the higher differentials of the spectral sequence vanish on the \( \text{HH}_*(R \otimes \mathbb{Z}/2\mathbb{Z}) \) factor. Thus when studying the spectral sequence differentials on the left side of (1.3.5), we know by naturality that their values on the \( \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) \) factor must be in the kernel of red, i.e. for \( k \geq 2 \)

\[
(1.3.6) \quad \text{im}(d_k|_{\text{HH}_*(R; \mathbb{Z}/2\mathbb{Z})}) \subset \ker(\text{red}) = e_3 \cdot \bar{\mathcal{A}}[e_4, e_8, \ldots] \otimes \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}).
\]

We will now see that for \( k = 3 \) this inclusion is actually an equality, which means that the spectral sequence differentials are as ‘destructive’ as they can be within the restriction of (1.3.6). If we study what is sitting in each coordinate of the spectral sequence, we will see that \( d_2|_{\text{HH}_*(R; \mathbb{Z}/2\mathbb{Z})} \) has nothing low enough to hit, given (1.3.6). \( d_3|_{\text{HH}_*(R; \mathbb{Z}/2\mathbb{Z})} \) already has something it could hit. \( R \otimes \mathbb{Z}/2\mathbb{Z} \) is spanned over \( \mathbb{Z}/2\mathbb{Z} \) by \( \zeta \), or more precisely the image of the \( \zeta \) chosen in (1.2) under the reduction mod 2 map. By the choice of \( \zeta \), its square vanishes under the reduction mod 2. Thus, by the well-known description of the Hochschild homology of an exterior algebra, the cycles \( 1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta \) and \( \zeta \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta \) in \( \text{HH}_k(R; \mathbb{Z}/2\mathbb{Z}) \cong \text{HH}_k(R \otimes \mathbb{Z}/2\mathbb{Z}) \) span it over \( \mathbb{Z}/2\mathbb{Z} \). For these generators we have
Lemma (1.4) \[ d_3(1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta) = e_3 \cdot (1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta) \]

\[ \text{Lemma (1.4)} \] \[ d_3(1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta) = e_3 \cdot (1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta). \]

Thus,

\[ \ker d_3|_{\text{HH}_*(R; \mathbb{Z}/2\mathbb{Z})} = \text{HH}_0(R; \mathbb{Z}/2\mathbb{Z}) \oplus \text{HH}_1(R; \mathbb{Z}/2\mathbb{Z}) \oplus \text{HH}_2(R; \mathbb{Z}/2\mathbb{Z}) \oplus \text{HH}_3(R; \mathbb{Z}/2\mathbb{Z}). \]

We recall the multiplicative structure of \( \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) \), as given by proposition (1.15) in [13]:

\[ \text{HH}_*(R; \mathbb{Z}/2\mathbb{Z}) \cong R/2R[e]/(e^2) \otimes \Gamma(a_2), \]

where \( \Gamma(a_2) \) is the divided power algebra spanned linearly over \( \mathbb{Z}/2\mathbb{Z} \) by elements \( a_2^{(i)} \), \( i \geq 0 \) (with \( a_2^{(0)} = 1 \) and \( a_2^{(1)} = a_2 \)) which multiply by the rule \( a_2^{(i)} \cdot a_2^{(j)} = (i+j) a_2^{(i+j)} \).

Following the proof there we see that in our case \( 1 \otimes \zeta \) represents the 1-dimensional class \( \epsilon \), and \( 1 \otimes \zeta \otimes \zeta \) the 2-dimensional class \( a_2 \) (and in general \( 1 \otimes \zeta^\otimes 2k \) the \( 2k \)-dimensional class \( a_2^{(k)} \)).

Using this notation, and the Leibniz rule again, we write

\[ \ker d_3 = \overline{A} \otimes R[e, a_2, e_3, e_4, e_8, \ldots]/(e^2, a_2^2, e_2^2) \]

\[ \oplus e_3 \cdot (a_2^{(2)}, a_2^{(4)}, a_2^{(8)}, \ldots) \cdot \overline{A} \otimes R[e, a_2, \ldots, e_4, e_8, \ldots]/(e^2, (a_2^{(i)})^2, e_i^2), \]

\[ \text{im} d_3 = e_3 \cdot \overline{A} \otimes R[e, a_2, a_2^{(2)}, a_2^{(4)}, \ldots, e_3, e_4, e_8, \ldots]/(e^2, (a_2^{(i)})^2, e_i^2) \]

and so

\[ E^4_{*,*} = \frac{\ker d_3}{\text{im} d_3} \cong \overline{A} \otimes R[e, a_2, e_4, e_8, \ldots]/(e^2, a_2^2, e_2^2). \]

Now in fact all the multiplicative generators of this last algebra have filtration degree smaller than four: 0 in the case of \( \overline{A} \otimes R \), 1 in the case of \( \epsilon \) and the \( e_i \), and 2 in the case of \( a_2 \). We deduce that all the \( d_k \) for \( k \geq 4 \) must vanish, and so

\[ E^\infty_{*,*} \cong \overline{A} \otimes R[e, a_2, e_4, e_8, \ldots]/(e^2, a_2^2, e_2^2). \]

Since we are working over a field, this is isomorphic to \( H^S_*(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \) as far as linear structure goes; they are also isomorphic as \( \overline{A} \) modules, since multiplication by \( \overline{A} \) involves only smashing the 0’th coordinate with some \( K(\mathbb{Z}, n) \) and adjusting dimensions-- an operation which preserves all filtration degrees. We still need to study \( H^S_*(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \)'s multiplicative structure in order to determine the action of the Bockstein operators on it, which is necessary for reading off the Eilenberg-MacLane spectra in \( \text{THH}(R) \).
**Terminology (1.6)** We will call a chain on some $\text{THH}^{(m)}(R)$ an unfiltered cycle if its boundary is zero in the singular chain complex of $\text{THH}^{(m)}(R)$ (as opposed to the associated filtered quotients). We will call a cycle with mod 2 coefficients absolute if it is the reduction of an integral cycle.

**Claim (1.7)** (Summary of facts from $[4]$) We can choose unfiltered homology classes $\bar{e}_4$, $\bar{e}_3$ on $\text{THH}(\mathbb{Z})$ such that $H^S_* (\text{THH}(\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A}[\bar{e}_3, \bar{e}_4]/(\bar{e}_3^2)$, with the mod 2 Bockstein operator sending $\beta_1(\bar{e}_4) = \bar{e}_3$. Moreover, we can pick them to be of a specific form which will be explained in section (3.4). Finally, reduction modulo 2, $\text{red} : \text{THH}(R) \rightarrow \text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z})$ satisfies $\text{red}_* (\bar{e}_4) = \lambda_1^2$, where $\lambda_1$ is the unfiltered class Bökstedt chose to represent in the $E^\infty$ term, for which $H^S_* (\text{THH}(\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A}[\lambda_1]$.

**Claim (1.8)** Since the inclusion homomorphism $\mathbb{Z} \hookrightarrow R$ sends Bökstedt’s $E^\infty$ classes $e_{2i}$ (which are represented by powers of $\bar{e}_4$) into classes we have used for $R$ with the same names, the image of his $\bar{e}_4$ (which we will call by the same name) in $H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$ and its powers will span representatives of all the $e_{2i}$ in (1.5.1). Since reduction mod 2 maps $\bar{e}_4$ to the class $\lambda_1^2$, which is algebraically independent of $\mathcal{A}$ in $H^S_* (\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$, $\bar{e}_4$ is algebraically independent of $\mathcal{A}$ in $H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$. We get that

$$\mathcal{A}[\bar{e}_4] \subset H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z}).$$

The elements of $R \otimes \mathbb{Z}/2\mathbb{Z}$ in $E^\infty_{0,0}$ can be represented by the 0-chains on $\text{THH}^{(0)}(R)$ corresponding to $r : S^0 \hookrightarrow R[S^0]$ sending the non-basepoint to $r$ times itself– this forms a 0-dimensional subcomplex closed under the multiplication $M$ and so its homology sits as a subalgebra of $H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$; this subalgebra satisfies no nontrivial relations with $\mathcal{A}[\bar{e}_4]$ because no such relations hold in the $E^\infty$ term. We can therefore say, in addition to (1.8.1), that

$$\mathcal{A} \otimes R[\bar{e}_4] \subset H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z}).$$

**Lemma (1.9)** The class $\bar{e}$ in (1.5.1) can be represented by an unfiltered class $\bar{e}$ which is absolute, i.e. $\beta_k(\bar{e}) = 0$ for all the Bockstein operators $\beta_k$, $k \geq 1$, and which satisfies $\bar{e}^2 = 0$.

**Lemma (1.10)** The class $a_2$ in (1.5.1) can be represented by an unfiltered class $\bar{a}_2$ with $\beta_1(\bar{a}_2) = (\zeta + a)\bar{e}$ (see (1.2))– $\zeta$ satisfies $R = \mathbb{Z}[\zeta]/(\zeta^2 + 2a\zeta + 2b)$, for which $\bar{e}_4 = \bar{a}_2^2$.

Now we know by (1.9) that in addition to (1.8.2), $H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$ contains a class $\bar{e}$ which is linearly independent of $\mathcal{A} \otimes R[\bar{e}_4]$ by the $E^\infty$ term and whose square is identified into $\mathcal{A} \otimes R[\bar{e}_4]$ as zero. And (1.10) tells us that beyond this copy of $\mathcal{A} \otimes R[\bar{e}, \bar{e}_4]/(\bar{e}^2) \subset H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$, we have a class $\bar{a}_2$, linearly independent of $\mathcal{A} \otimes R[\bar{e}, \bar{e}_4]/(\bar{e}^2)$ by the $E^\infty$ term, whose square is identified to $\bar{e}_4$. The algebra formed by all of these accounts for all of $H^S_* (\text{THH}(R); \mathbb{Z}/2\mathbb{Z})$, by dimension counting in the $E^\infty$ term. Since $\bar{a}_2$ satisfies no nontrivial linear relation with $\mathcal{A} \otimes R[\bar{e}, \bar{e}_4]/(\bar{e}^2)$ and $\bar{e}_4$ is algebraically independent of $\mathcal{A} \otimes R[\bar{e}]/(\bar{e}^2)$, $\bar{a}_2$ is algebraically independent of $\mathcal{A} \otimes R[\bar{e}]/(\bar{e}^2)$. We thus deduce
Corollary (1.11) \( H^S_*(\mathbf{THH}(R); \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{A} \otimes R[\bar{\epsilon}, \bar{a}_2]/(\bar{\epsilon}^2) \).

If we look at the result (1.11) to find generators of \( H^S_*(\mathbf{THH}(R); \mathbb{Z}/2\mathbb{Z}) \) over \( \mathcal{A} \), we find two absolute classes, which can be taken to be \( \iota \) and \( \zeta \), in dimension 0 (these two are linearly independent since \( \iota \)'s invertibility in \( R \otimes \mathbb{Z}/2\mathbb{Z} \) means that it is equal either to 1 or to \( 1 + \zeta \)). Alternatively, these can be taken to be 1 and \( \zeta \). In addition to this, for any \( 1 \leq \ell \) we have the products of \( \bar{a}_2^\ell \) and \( \bar{a}_2^{\ell-1} \bar{\epsilon} \) with the 0-dimensional generators. To determine the 2-torsion Eilenberg-MacLane spectra which appear in \( \mathbf{THH}(R) \), we need to look at the way the Bockstein operators act on these generators. We know from (1.10) that

\[
\beta_1(\bar{a}_2) = (\zeta + a)\bar{\epsilon} \quad \beta_1(\zeta\bar{a}_2) = (\zeta^2 + a\zeta)\bar{\epsilon} = -(2b + a\zeta)\bar{\epsilon} \equiv 0
\]

in the case of \( a \) even, but

\[
\beta_1(\zeta\bar{a}_2) = (\zeta^2 + a\zeta)\bar{\epsilon} = -(2b + a\zeta)\bar{\epsilon} \equiv \zeta\bar{\epsilon}
\]

in the case of \( a \) odd. For even \( a \), the calculation used in the proof of (1.10) for showing the calculation above (formula (3.6.1)) works also with coefficients in \( \mathbb{Z}/8\mathbb{Z} \), from which we can deduce

\[
\beta_2(\bar{a}_2) = \zeta \bar{\epsilon}.
\]

Lemma (1.12) \( \beta_1(\bar{a}_2^2) = 0 \) but \( \beta_2(\bar{a}_2^2) = (\zeta + a)\bar{a}_2\bar{\epsilon} \).

As before, this implies in the case of odd \( a \) that \( \beta_2(\zeta\bar{a}_2^2) \equiv \zeta\bar{a}_2\bar{\epsilon} \), and in the case of even \( a \) that for \( i = 1, 2 \), \( \beta_i(\zeta\bar{a}_2^2) = 0 \), but \( \beta_3(\zeta\bar{a}_2^2) = i\bar{a}_2\bar{\epsilon} \). For higher powers, we have part (vi) of proposition 1.5 in [16], which tells us that for an even-dimensional mod 2 homology class \( y \) on an infinite loop space, if \( \beta_i(y) = 0 \) for \( 1 \leq i < k \) (with \( k > 1 \)) we have \( \beta_i(y^2) = 0 \) for \( 1 \leq i \leq k \) and \( \beta_{k+1}(y^2) = y\beta_k(y) \). This will inductively give us the behavior of the higher Bockstein operators defined on the classes \( (\bar{a}_2)^{2^\alpha} \), and from them we can as before deduce the behavior on the multiples of these classes by \( \zeta \). For powers \( (\bar{a}_2)^\ell \), with \( \ell = 2^\alpha \) and \( \beta \) odd, we observe that \( (\bar{a}_2)^\ell = \left((\bar{a}_2)^{\frac{\beta-1}{2}}\right)^{2^{\alpha+1}}(\bar{a}_2)^{2^\alpha} \), and applying the Leibniz rule to products of a lifting of \( \bar{a}_2 \) to a chain with appropriate coefficients, we get

Claim (1.13) \( H^S_*(\mathbf{THH}(R); \mathbb{Z}/2\mathbb{Z}) \) is generated over \( \mathcal{A} \) by two 0-dimensional generators, in addition to generators \((\bar{a}_2)^\ell, \zeta(\bar{a}_2)^\ell, (\bar{a}_2)^{\ell-1}\bar{\epsilon}, \) and \( \zeta(\bar{a}_2)^{\ell-1}\bar{\epsilon} \) for any \( 1 \leq \ell \). In the case that \( a \) in \( \zeta \)'s minimal polynomial (see (1.2)) is even, then if \( \ell = 2^\alpha \beta \) for \( \beta \) odd, these are related by \( \beta_{\alpha+1}((\bar{a}_2)^\ell) = \zeta(\bar{a}_2)^{\ell-1}\bar{\epsilon} \) and \( \beta_{\alpha+2}(\zeta(\bar{a}_2)^\ell) = \zeta(\bar{a}_2)^{\ell-1}\bar{\epsilon} \). In the case that \( a \) is odd, they are related by \( \beta_{\alpha+1}((\bar{a}_2)^\ell) = (\zeta + 1)(\bar{a}_2)^{\ell-1}\bar{\epsilon} \) and \( \beta_{\alpha+1}(\zeta(\bar{a}_2)^\ell) = \zeta(\bar{a}_2)^{\ell-1}\bar{\epsilon} \) and we get

We get
Theorem (1.14) Let $R$ be the ring of integers in a quadratic extension of the rationals which ramifies at 2; for a suitable $\zeta \in R$, $R = \mathbb{Z}[\zeta]/(\zeta^2 + 2a\zeta + 2b)$, $a, b \in \mathbb{Z}$. Then if $a$ is even, $\text{THH}(R)$ consists of

$$K(\mathbb{Z}, 0)^2 \times \prod_{\ell=1}^{\infty} \left( K(\mathbb{Z}/2^{\alpha+1}\mathbb{Z}, 2\ell - 1) \times K(\mathbb{Z}/2^{\alpha+2}\mathbb{Z}, 2\ell - 1) \right)$$

and $p$-torsion spectra for $p \neq 2$, while if $a$ is odd, it consists of

$$K(\mathbb{Z}, 0)^2 \times \prod_{\ell=1}^{\infty} ' \left( K(\mathbb{Z}/2^{\alpha+1}\mathbb{Z}, 2\ell - 1) \times K(\mathbb{Z}/2^{\alpha+1}\mathbb{Z}, 2\ell - 1) \right)$$

and $p$-torsion spectra for $p \neq 2$.

§2. Concluding the Calculation– Unramified Extensions

In the previous section, while looking at rings of integers in quadratic extensions of the rationals which are ramified at the prime 2, we have not discussed those which do not ramify at 2, for which the calculation of $\text{THH}$ is much simpler. This calculation is given in the following two lemmas. For rings of integers in quadratic extensions which ramify only at $p = 2$, these lemmas will conclude the calculation of the stable homotopy type of the spectrum $\text{THH}(R)$.

Lemma (2.1) Let $R$ be the ring of integers in an extension of the rationals which does not ramify at a prime $p$. Then $\text{HH}_0(R; \mathbb{Z}/p\mathbb{Z}) \cong R \otimes \mathbb{Z}/p\mathbb{Z}$ and $\text{HH}_*(R; \mathbb{Z}/p\mathbb{Z}) \cong 0$ for $* > 0$.

Proof. The fact that $\text{HH}_0(R; \mathbb{Z}/p\mathbb{Z}) \cong R \otimes \mathbb{Z}/p\mathbb{Z}$ follows from $R$'s commutativity, since commutativity makes the first differential in the standard bar complex calculating Hochschild homology into the zero map. For the second fact, we know by the flatness of $\mathbb{Z}_p$ that $\text{HH}_*(\widehat{R}_p/\mathbb{Z}_p) \cong \text{HH}_*(R) \otimes \mathbb{Z}_p$. Now by assumption, the quotient field of $\widehat{R}_p$ is an unramified extension of $\mathbb{Q}_p$, and so by Lemma (1.4) in [13], $\text{HH}_*(\widehat{R}_p/\mathbb{Z}_p) = 0$ for $* > 0$. This tells us that $\text{HH}_*(R)$ consists, in positive dimensions, entirely of $q$-torsion for $q \neq p$, from which we get by the universal coefficient theorem (and the fact that $\text{HH}_0(R) \cong R$ is torsion-free) that $\text{HH}_*(R; \mathbb{Z}/p\mathbb{Z}) \cong 0$ for $* > 0$. □

Lemma (2.2) If $R$ is an extension of the integers which is additively a free $\mathbb{Z}$-module of rank $k$, and if $\text{HH}_0(R; \mathbb{Z}/p\mathbb{Z}) \cong R \otimes \mathbb{Z}/p\mathbb{Z}$ and $\text{HH}_*(R; \mathbb{Z}/p\mathbb{Z}) \cong 0$ for $* > 0$, then $H^S_*(\text{THH}(R); \mathbb{Z}/p\mathbb{Z}) \cong R \otimes H^S_*(\text{THH}(\mathbb{Z}); \mathbb{Z}/p\mathbb{Z})$ as a $\mathbb{A}$-algebra with action of the Bockstein operator, and higher Bockstein operators when they are defined (and as a result of this, the $p$-torsion Eilenberg-MacLane spectra in the decomposition of $\text{THH}(R)$ consists of exactly $k$ copies of what we get for $\text{THH}(\mathbb{Z})$).

Proof. By (1.0.1),

$$E^2_{*,*}(\text{THH}(R); \mathbb{Z}/p\mathbb{Z}) \cong \text{HH}_*(R; \mathbb{Z}/p\mathbb{Z}) \otimes E^2_{*,*}(\text{THH}(\mathbb{Z}); \mathbb{Z}/p\mathbb{Z})$$

$$\cong (R \otimes \mathbb{Z}/p\mathbb{Z}) \otimes E^2_{*,*}(\text{THH}(\mathbb{Z}); \mathbb{Z}/p\mathbb{Z})$$
and the copy of $R \otimes \mathbb{Z}/p\mathbb{Z}$ sits in the coordinate $(0,0)$ where no nontrivial differentials can map from it or into it. Therefore this $R \otimes \mathbb{Z}/p\mathbb{Z}$ lives on to the $E^\infty$ term. The splitting $E^\infty_{*,*}(\text{THH}(R); \mathbb{Z}/p\mathbb{Z}) \cong (R \otimes \mathbb{Z}/p\mathbb{Z}) \otimes E^\infty_{*,*}(\text{THH}(\mathbb{Z}); \mathbb{Z}/p\mathbb{Z})$ implies a similar splitting of stable homology since it tells us that the subalgebra $R \otimes \mathbb{Z}/p\mathbb{Z}$ satisfies no relations with the rest of the homology, beyond the obvious relations in a tensor product. Since the classes in this $R \otimes \mathbb{Z}/p\mathbb{Z}$ can be represented by absolute cycles, multiplication by them commutes with any Bockstein operators that are defined. □

By [4], the $p$-torsion in $\text{THH}(\mathbb{Z})$ consists of a copy of $K(\mathbb{Z}/p^a\mathbb{Z}, 2p^a b - 1)$ for each $b$ coprime to $p$ and $a \geq 1$. Using the last lemma, and the result of (1.14), we conclude

Theorem (2.3) Let $R$ be the ring of integers in a quadratic extension of the rationals which ramifies only at 2, i.e. $R = Z[i]$ or $R = Z[\sqrt{\pm 2}]$.

$$\text{THH}(Z[i]) \cong K(Z, 0)^2 \times \prod_{j=1}^{\infty} K(Z/2jZ, 2j - 1)^2$$

$$\text{THH}(Z[\sqrt{\pm 2}]) \cong K(Z, 0)^2 \times \prod_{j=1}^{\infty} (K(Z/2jZ, 2j - 1) \times K(Z/4jZ, 2j - 1)).$$

§3. Proofs of the Technical Lemmas

(3.1) The Definition of THH and Bökstedt’s Spectral Sequence We will start the presentation of the technical proofs by reviewing Bökstedt’s definition of $\text{THH}$ from [3] and [4] (with the indexing of section 1.7 of [12], and the notations of [14]). Let $R$ be an associative ring with unit. For a finite simplicial object $X_\bullet$, we will use the notation $R[[X_\bullet]]$ to denote the geometric realization of the simplicial object given in each degree by the free module $RX_n = R[X_n]/R*$, with $*$ the degenerate simplex at the basepoint. Let $1_{[X_\bullet]} : [X_\bullet] \to R[[X_\bullet]]$ be the map induced by $\alpha \mapsto 1 \cdot \alpha$, and for any two such objects $[X_\bullet]$ and $[Y_\bullet]$, let

$$\mu_{[X_\bullet],[Y_\bullet]} : R[[X_\bullet]] \wedge R[[Y_\bullet]] \to R[[X_\bullet] \wedge [Y_\bullet]]$$

be the map induced by $R$'s multiplication. For $m \geq 0$, we set

$$(3.1.1) \quad T_r^{(m)} = \text{hocolim}_{(j_0, j_1, \ldots, j_r) \in \binom{I^m}{r+1}} (S^{j_0} \wedge S^{j_1} \wedge \cdots \wedge S^{j_r}, R[S^{j_0}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_r}] \wedge S^m),$$

where $I$ is the category of integers $n = \{1, 2, \ldots, n\}$, for $n \geq 0$, and injective maps between them, and $I^m$ is the $m$-fold cartesian product of $I$ with itself. $S^{(0, \emptyset, \ldots, \emptyset)}$ is defined to be $S^0$, and for an element $j = (n_1, n_2, \ldots, n_m) \in I^m$ we define $S^j$ as a smash product of spheres $S^{j_1}$ indexed by the elements of all the non-empty sets $n_i$ in $j$. Thus topologically each $S^j$ is a sphere whose dimension is the sum of cardinalities of all the coordinates of $j$, and each $R[S^j]$ is the Eilenberg-MacLane space of $R$ (as an additive group) in that dimension. We will use integers $n$ to denote objects of the form $(n, \emptyset, \emptyset, \ldots, \emptyset) \in I^m$. Topologically,
for $m > 0$, $T_r^{(m)}$ is homotopy equivalent to the space we would have gotten had we taken the homotopy colimit to run over $I$ alone; the reason for the more complicated category is to facilitate the definition of a product the topological Hochschild homology spectrum. For $m = 0$, the definition makes sense if we interpret $I^0$ as the category consisting of the empty set; this will be convenient for the easier of the simplicial calculations which follow. We define cyclic simplicial boundary maps $d_i : T_r^{(m)} \to T_r^{(m)}$ for $0 \leq i \leq r$ which ‘multiply’ the $i$th and $(i + 1)$st ($r$th and $0$th, in the case $i = r$) coordinates: we identify $S^{j_i} \land S^{j_{i+1}}$ with $S^{j_i+j_{i+1}}$, and compose the maps in $T_r^{(m)}$ with

$$\mu_{S^{j_i}, S^{j_{i+1}} : R[S^{j_i}] \land R[S^{j_{i+1}}] \to R[S^{j_i} \land S^{j_{i+1}}] = R[S^{j_i+j_{i+1}}]$$

in the $i$’th coordinate. When $i = r$ we do this treatment in the $0$’th coordinate, using $\mu_{S^{j_r}, S^{j_0}}$. The $+$ operation on $I^m$ used here is concatenation in each coordinate,

$$+ : ((n_1, n_2, \ldots, n_m), (n'_1, n'_2, \ldots, n'_m)) \mapsto (n_1 + n'_1, n_2 + n'_2, \ldots, n_m + n'_m).$$

Degeneracy maps $s_i : T_r^{(m)} \to T_r^{(m+1)}$ for $0 \leq i \leq r$ are defined by inserting the maps $1_{S^{(0,0,\ldots,0)}} : S^{(0,0,\ldots,0)} \to R[S^{(0,0,\ldots,0)}]$ after the $i$’th coordinate.

For each $r$, spectrum structure maps $\Sigma T_r^{(m)} \to T_r^{(m+1)}$ are defined as in [14], except that here we also use the embedding $I^m \hookrightarrow I^{m+1}$ given by $(n_1, \ldots, n_m) \mapsto (n_1, \ldots, n_m, \emptyset)$ to induce a map of the homotopy colimits. These structure maps make $\{m \mapsto T_r^{(m)}\}$ for $m \geq 1$ into an $\Omega$-spectrum. For each $m$, $\{r \mapsto T_r^{(m)}\}$ is a simplicial space whose realization is called $THH^{(m)}(R)$. Since the simplicial structure maps commute with the spectrum structure maps, the latter induce spectrum structure maps on the realizations, and the topological Hochschild homology spectrum $\text{THH}(R)$ is the resulting spectrum $\{m \mapsto \text{THH}^{(m)}(R)\}$. Note that Bökstedt’s original indexing convention, with the category $I$ as the fixed limit category in (3.1.1), gives a very different $\text{THH}^{(0)}(R)$ than the one we get here (specifically, it gives the loopspace of $\text{THH}^{(1)}(R)$). If we want to think of $\text{THH}(R)$ as an $\Omega$-spectrum, we should look at $\{m \mapsto \text{THH}^{(m)}(R)\}$ for $m \geq 1$, but we will use $\text{THH}^{(0)}(R)$ in those calculations which already work over it, since it is a much simpler space—the geometric realization of the standard bar complex for Hochschild homology.

The filtration by simplicial ‘strata’ on each $\text{THH}^{(m)}$ gives a filtration of the singular chains on it, which can be used for a spectral sequence calculating its homology with coefficients in $\mathbb{Z}/p\mathbb{Z}$. For each $\text{THH}^{(m)}$ the $E^1$ term of this spectral sequence will have $E^1_{r,s} \cong H_* (T_r^{(m)}; \mathbb{Z}/p\mathbb{Z})$. The spectrum suspension maps respect this filtration, and therefore they induce maps of the associated spectral sequences. On the $E^1$-term, the spectrum suspension maps send

$$E^1_{r,s}(\text{THH}^{(m)}(R); \mathbb{Z}/p\mathbb{Z}) \to E^1_{r,s+1}(\text{THH}^{(m+1)}(R); \mathbb{Z}/p\mathbb{Z}),$$

and for any fixed $r$ and $s$ these maps stabilize eventually, to become isomorphisms from some point on. Let $K(R, n)$ denote the Eilenberg-MacLane spectrum $\{m \mapsto K(R, n+m)\},$
where \( K(R, n) \) is the Eilenberg-MacLane space corresponding to \( R \)’s additive group (of which \( R[S^n] \) is a model). Then as spectra, \( \{ m \mapsto T_r^{(m)} \} \simeq K(R, 0)^{\wedge (r+1)} \), and so

\[
\lim_{m \to \infty} E^1_{r, \ast + m}(\text{THH}^{(m)}(R); \mathbb{Z}/p\mathbb{Z}) \cong H^S_\ast(\{m \mapsto T_r^{(m)}\}; \mathbb{Z}/2\mathbb{Z}) \cong H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z})^\otimes r + 1.
\]

The dimension-shifting maps on the spectral sequences above commute with boundary maps, and therefore they commute with the spectral sequence differentials. Because of this, the spectrum suspensions induce (dimension shifting) homomorphisms on the \( E^k \) terms of the spectral sequence for all \( k > 1 \). So we can work with the limit spectral sequence, which calculates \( H^S_\ast(\text{THH}(R); \mathbb{Z}/p\mathbb{Z}) \). The spectral sequence differential calculations can be carried on in any \( \text{THH}^{(m)} \) in which the class we want to work with is already represented, and we will use this often.

The \( E^1 \)-term of the limit spectral sequence is

\[
E^1_{r, \ast} \cong H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z})^\otimes r + 1
\]

The spectral sequence \( d_1 \) is exactly the Hochschild homology differential (for graded algebras), and so we also have

\[
E^2_{r, \ast} \cong \text{HH}_r(H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z}))
\]

(in the graded sense). Bökstedt showed in [4] that this spectral sequence collapses at \( E^p \) for \( R = \mathbb{Z} \) and \( R = \mathbb{Z}/p\mathbb{Z} \). We note that if \( R \) is additively a free \( \mathbb{Z} \)- or \( \mathbb{Z}/p\mathbb{Z} \)-module, then

\[
H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z}) \cong H^S_\ast(K(\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z}) \otimes R
\]

or

\[
H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z}) \cong H^S_\ast(K(\mathbb{Z}/p\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z}) \otimes R,
\]

respectively, and so, by a Künneth formula argument using shuffle products with signs adjusted to respect the graded structure, we get formulae (1.0.1) and (1.0.2),

\[
E^2_{s, \ast} \cong \text{HH}_s(H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z})) \cong \text{HH}_s(H^S_\ast(K(\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z})) \otimes \text{HH}_s(R; \mathbb{Z}/p\mathbb{Z})
\]

and

\[
E^2_{s, \ast} \cong \text{HH}_s(H^S_\ast(K(R, 0); \mathbb{Z}/p\mathbb{Z})) \cong \text{HH}_s(H^S_\ast(K(\mathbb{Z}/p\mathbb{Z}, 0); \mathbb{Z}/p\mathbb{Z})) \otimes \text{HH}_s(R),
\]

respectively.
(3.2) Notations. The proofs of the technical lemmas are done by simplicial calculations. Since these are much easier to write when we look at the smash products $R[S^{j_0}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$ themselves, rather than on the looped-down versions, we will work on the delooped $\text{THH}^{(m)}$'s for different values of $m$ as necessary. As explained when setting up Bökstedt’s spectral sequence, the spectrum suspension maps induce dimension-lifting homomorphisms of the $E^1$-terms of the respective spectral sequences, which commute with the spectral sequence differentials $d_i$, $i \geq 1$. The maps induced by multiplication commute with the spectral sequence suspension maps too, since the multiplication itself homotopy-commutes with the spectrum suspension maps. So we can carry out our calculations of differentials and products on whichever $\text{THH}^{(m)}$ we please, as long as the classes we are interested in are defined on it. Since in our case we are interested in the Hochschild homology classes in $(1.0.1)$ which are already images of classes in the spectral sequence for $\text{THH}^{(0)}(R)$ or $\text{THH}^{(1)}(R)$, this is not a problem. More specifically, elements representing Hochschild homology classes in $\text{HH}_q(R; \mathbb{Z}/2\mathbb{Z})$ can be given as sums of monomials $a_0 \otimes a_1 \otimes \cdots \otimes a_q$ with $a_i \in R$. In [14], it was sufficient to represent each such monomial by the $q$-simplex corresponding to the map $(S^0)^{\wedge (q+1)} \to R[S^0]^{\wedge (q+1)}$ which sends the non-basepoint to the point $(a_0, a_1, \ldots, a_q)$, because if the monomials were chosen wisely, the sum of the $q$-simplices corresponding to them was an unfiltered cycle. So we were looking at $q$-dimensional cycles on $\text{THH}(R)^{(0)}$ which came from 0-dimensional cycles on $T_q^{(0)}$. Here the situation is more complicated: we will, in general, want to represent each $a_0 \otimes a_1 \otimes \cdots \otimes a_q$ by the $(q+1)$-dimensional chain on $R[S^1]^{\wedge (q+1)}$ which will, in the notation defined below, be called $a_0 \wedge a_1 \wedge \cdots \wedge a_q$ (or sometimes also by chains which will be called $a_0 \wedge a_1 \wedge \cdots \wedge a_q$ in the notation below, on $R[S^0] \wedge R[S^1]^{\wedge q}$). These will not, in general, be unfiltered cycles, but we will explicitly complete them with lower filtration elements in order to calculate the spectral sequence differentials on them. In effect what we are doing is building a subcomplex of some $\text{THH}^{(m)}(R)$ where the homology class in question is represented in the $E^2$ term, and doing the calculation on the spectral sequence associated to that subcomplex. The fundamental mechanism will be to identify $r$-dimensional chains on $R[S^{j_0}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$ as $(r + k)$-dimensional chains on $\text{THH}^{(m)}(R)$, where $m$ is the sum of cardinalities of all the coordinates of all the $j_a$’s. This will work by identifying $R[S^{j_0}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$ into $T_k^{(m)}$ via

$$
\phi: R[S^{j_0}] \wedge \cdots \wedge R[S^{j_k}] \to (S^{j_0} \wedge \cdots \wedge S^{j_k}, R[S^{j_0}] \wedge \cdots \wedge R[S^{j_k}] \wedge S^{j_0} \wedge \cdots \wedge S^{j_k}),
$$

$$
x \mapsto (\theta \mapsto (x, \theta), \quad \forall \theta \in S^{j_0} \wedge \cdots \wedge S^{j_k}).
$$

We are, of course, using the fact that $S^{j_0} \wedge \cdots \wedge S^{j_k}$ is topologically an $S^m$. We regard this sphere as a smash product of circles $S^1$ indexed by the disjoint union of the $S^{j_a}$. Once we have identified $R[S^{j_0}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$ into $T_k^{(m)}$ via $\phi$, it is clear how any $r$-dimensional chain $C$ on it corresponds to an $(r + k)$-dimensional chain $\phi_*(C) \times \Delta^k$ on $\text{THH}^{(m)}(R)$. For brevity’s sake we will, however, omit writing $\phi$ from now on, and refer to this $(r + k)$-chain on $\text{THH}^{(m)}(R)$ as $C \times \Delta^k$. Our indexing of $S^m$ is an ad-hoc one created for a particular chain, but if we make the identification for a particular chain on $T_k^{(m)}$ we can make a compatible identification for the images of the chain under the simplicial space face maps. Essentially what we are doing is embedding a complex, representing the
homology class in question, into a fixed $\text{THH}(R)^{(m)}$, and this complex is constructed step by step from liftings of the spectral sequence differentials. The simplicial space boundary maps can, with this understanding, simply be viewed as applying the smash maps on the appropriate coordinates for chains on $R[S^{2h}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$. When we look at the last simplicial boundary map, the wraparound term, there can be a correction of sign needed – for most of the calculations here, this does not matter, since the coefficients are in $\mathbb{Z}/2\mathbb{Z}$, but since the suspension coordinate $S^m$ is not involved and the chains discussed are images of chains from $(S^{2h}, R[S^{2h}]) \wedge (S^{j_1}, R[S^{j_1}]) \wedge \cdots \wedge (S^{j_k}, R[S^{j_k}])$ (appropriately suspended), the sign correction is the ‘stable dimension’ of the chain in the last coordinate times the sum of the ‘stable dimensions’ of the chains it is pushed in front of.

We will abbreviate the notation for Bökstedt’s basic FSP multiplication maps $\mu_{[X,|.|Y,|.|}$, and say that for any $j, j' \in I^m$, $\mu$ denotes $\mu_{S^j, S^{j'}} : R[S^j] \wedge R[S^{j'}] \to R[S^{j+j'}]$ and also the map this induces on simplicial chains. Moreover, since the $\mu_{[X,|.|Y,|.|}$ are strictly associative, we will use $\mu$ also to denote iterated applications of $\mu$ from $R[S^{2h}] \wedge R[S^{j_1}] \wedge \cdots \wedge R[S^{j_k}]$ to $R[S^{2h+j_1+\cdots+j_k}]$ and the map they induce on simplicial chains.

For any $(r+k)$-dimensional chain on $\text{THH}^{(m)}(R)$ which is of the form $C \times \Delta^k$ with $C$ an $r$-dimensional chain on $T_k^{(m)}$, its boundary has, by the Leibniz formula, two components; since we are working with coefficients in $\mathbb{Z}/2\mathbb{Z}$, signs are irrelevant and we write that the boundary

$$\partial(C \times \Delta^k) = b(C \times \Delta^k) + d(C \times \Delta^k)$$

with $b(C \times \Delta^k) = (\partial C) \times \Delta^k$ corresponding to the boundary map of simplicial chains on $T_k^{(m)}$, and $d(C \times \Delta^k) = \sum_{i=0}^k d_i(C) \times \Delta^{k-1}$ corresponding to the simplicial space structure maps.

For $a \in R$, we let $a$ denote the 0-dimensional chain $1 \cdot (a \cdot (\text{the non–basepoint}))$ on $R[S^0] = R[S^0, \ldots, \emptyset]$. If we take a simplicial model of $S^1$ with one non-degenerate 1-simplex $\alpha$, we let $\underline{a}$ denote the 1-dimensional chain $1 \cdot a \alpha$ on $R[S^1] = R[S^{1,\emptyset,\ldots,\emptyset}]$. Note that with this notation, we have for any $a_1, a_2 \in R$, $\mu(a_1 \wedge a_2) = \mu(a_1 \wedge a_2) = a_1 a_2$. We also set $\underline{a} = \mu(a \wedge 1) = \mu(\underline{1} \wedge a)$, and deduce $\mu(a_1 \wedge a_2) = \mu(a_1 \wedge a_2) = a_1 a_2$ for any $a_1, a_2 \in R$. More generally, for any two chains $C_1, C_2$ on some $R[S^{j_1}], R[S^{j_2}]$ and any $a \in R$ we have

$$\mu(\mu(C_1 \wedge a) \wedge C_2) = \mu(C_1 \wedge a \wedge C_2),$$

and since $R$ is commutative we also have

$$\mu(C \wedge a) = \mu(a \wedge C)$$

for any chain $C$ on $R[S^j]$.
(3.3) Proof of Lemma (1.4). We represent $1 \otimes \zeta \otimes \cdots \otimes \zeta$ by the chain $1 \wedge \zeta \wedge \cdots \wedge \zeta \times \Delta^k$, as explained above, and proceed to trace its differentials. We have

$$d(1 \wedge \zeta^k \times \Delta^k) = \sum_{j=0}^{k-2} 1 \wedge \zeta^j \wedge 2^{k-2-j} \times \Delta^{k-1} \wedge \zeta^{k-2-j} \times \Delta^{k-1}.$$ 

We define

$$(3.3.1) M_1 = (s_0(\alpha) + s_1(\alpha)),$$

a 2-simplex in the model of $R[S^1]$ defined from the simplicial structure we have chosen for $S^1$. $M_1$’s boundary is precisely $1 + 1 + 2$. Letting

$$L_1 = \sum_{j=0}^{k-2} 1 \wedge \zeta^j \wedge \mu(M_1 \wedge \zeta) \wedge \zeta^{k-2-j} \times \Delta^{k-1},$$

we observe that

$$b(L_1) = d(1 \wedge \zeta^k \times \Delta^k),$$

while

$$d(L_1) = \left( \mu(1 \wedge M_1 \wedge \zeta) \wedge \zeta^k \wedge \zeta^k \wedge \zeta \times \Delta^k \right) \times \Delta^{k-2}.$$ 

We consider the 3-chain $\mu(1 \wedge M_1) + \mu(M_1 \wedge 1)$ on $R[S^2]$. It is a cycle, even when we look at it as a chain with integral coefficients. But we know that $H_3(Z[S^2]; Z) \cong H_3(\mathbb{CP}^{\infty}; Z) \cong \mathbb{Z}$, and of course our $R$ is additively just a direct sum of copies of $Z$, so there exists a 4-dimensional chain $M_2$ on $R[S^2]$ whose boundary is $\mu(1 \wedge M_1) + \mu(M_1 \wedge 1)$. We reduce coefficients mod 2 and obtain a chain which we will also call $M_2$. Setting

$$L_2 = \left( \mu(M_2 \wedge \zeta) \wedge \zeta^k \wedge \zeta \times \Delta^k \right) \times \Delta^{k-2}.$$ 

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We see that
\[ b(L_2) = d(L_1); \]
in particular this shows that the spectral sequence differential \( d_2[1 \wedge \zeta^k \times \Delta^k] = [d(L_1)] = 0 \) in the \( E^2 \)-term, a fact which we already know for dimension reasons and the \( R \otimes \mathbb{Z}/2\mathbb{Z} \) calculation in [14]. We let
\[
L = \sum_{j=0}^{k-6} \sum_{\ell=0}^{j} \sum_{m=0}^{\ell} 1 \wedge \zeta^m \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{\ell-m} \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{j-\ell} \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{k-6-j} \\
+ \sum_{j=0}^{k-5} \sum_{\ell=0}^{j} 1 \wedge \zeta^{\ell} \wedge \mu(M_2 \wedge \underline{\nu} \zeta) \wedge \zeta^{j-\ell} \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{k-5-j} \\
+ \sum_{j=0}^{k-5} \sum_{\ell=0}^{j} 1 \wedge \zeta^{\ell} \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{j-\ell} \wedge \mu(M_2 \wedge \underline{\nu} \zeta) \wedge \zeta^{k-5-j} \\
+ \sum_{j=0}^{k-4} \mu(M_2 \wedge \underline{\nu}) \wedge \zeta^{j} \wedge \mu(M_1 \wedge \underline{\nu}) \wedge \zeta^{k-4-j} \times \Delta^{k-3}.
\]
Comparing,
\[
d(L_2) + b(L) = \sum_{j=0}^{k-4} (1 \wedge \zeta^j \wedge \mu(\zeta \wedge M_2 \wedge \underline{\nu} \zeta) \wedge \zeta^{k-4-j} \\
+ 1 \wedge \zeta^j \wedge \mu(M_2 \wedge \underline{\nu} \zeta^2) \wedge \zeta^{k-4-j} \\
+ 1 \wedge \zeta^j \wedge \mu(M_1 \wedge \underline{\nu} \wedge M_1 \wedge \underline{\nu}) \wedge \zeta^{k-4-j} \times \Delta^{k-3}.
\]
Now the sum
\[
(3.3.2) \quad \sigma(4,0,0,...,0) = \mu(\zeta \wedge M_2 \wedge \underline{\nu} \zeta) + \mu(M_2 \wedge \underline{\nu} \zeta^2) + \mu(M_1 \wedge \underline{\nu} \wedge M_1 \wedge \underline{\nu})
\]
is a 6-dimensional cycle on \( R[S^4] = R[S(4,0,0,...,0)] \) as it sits in \( T_{k-4}^{(k+1)} \) in the formula above. We want to relate it to the 6-dimensional cycle \( \sigma(0,...,0,4,0,0,0) \) on \( R[S(0,...,0,4,0,0,0)] \) which has the same structure as \( \sigma_{k-4} \), but over the differently indexed sphere. The homotopy colimit will make both of these cycles homologous to cycles on \( R[S(4,...,0,4,0,0,0)] \), which are their suspensions in the respective complementary four coordinates. And since these cycles will represent the same homology class, their images must be homologous and therefore we see that there is a chain \( h \) on \( T_{k-4}^{(k+1)} \) such that \( b(h) = \sigma_{k-4} = \sigma(4,0,0,...,0) + \sigma(0,...,0,4,0,0,0) \). We set
\[
L_h = \sum_{j=0}^{k-4} (1 \wedge \zeta^j \wedge h \wedge \zeta^{k-4-j}) \times \Delta^{k-3}
\]
and get
\[
    d(L_2) + b(L + L_h) = \sum_{j=0}^{k-4} (1 \wedge \zeta^j \wedge s(0,\ldots,0,4,\emptyset,\emptyset,\emptyset) \wedge \zeta^{k-4-j}) \times \Delta^{k-3}
\]

\[
= M_* \left( \left( (1 \wedge \zeta^{k-4}) \times \Delta^{k-4} \right) \times \left( (1 \wedge s(4,\emptyset,\emptyset,\emptyset)) \times \Delta^1 \right) \right)
\]

where \(s(4,\emptyset,\emptyset,\emptyset)\) is the same as \(s(4,\emptyset,\emptyset,\ldots,\emptyset)\) except for having fewer extra indices of \(\emptyset\), and \(M\) is our multiplication map of \((1.3.3)\), mapping \(THH^{(k-3)} \times THH^{(4)} \to THH^{(k+1)}\) in this case. This is because as far as non-degenerate simplices are concerned, \(M\) is a shuffle product, concatenating indexing coordinates \(I\). This is because as far as non-degenerate simplices are concerned, \(M\) is a shuffle product, concatenating indexing coordinates \(I\).

Thus, since as far as non-degenerate simplices are concerned, \(M\) is a shuffle product, concatenating indexing coordinates \(I\), we get that
\[
\text{red}_*(s(4,\emptyset,\emptyset,\emptyset)) = \text{red}_*\mu(M_1 \wedge \zeta \wedge M_1 \wedge \zeta) = \text{red}_*\mu\left( \mu(M_1 \wedge 1) \wedge \mu(M_1 \wedge 1) \right) \wedge \iota^2.
\]

For the 2-simplex \(M_1\) that we have chosen, \(\text{red}_*(M_1) = \xi_1 \in H_2((R \otimes \mathbb{Z}/2\mathbb{Z})[S^1]; \mathbb{Z}/2\mathbb{Z})\), and since smashing with the unit sphere \(\mathbb{1}\) is our suspension map \(R[S^k] \to R[S^{k+1}]\) which sends stable classes to their suspensions which have the same name, \(\text{red}_*(\mu(M_1 \wedge 1)) = \xi_1 \in H_3((R \otimes \mathbb{Z}/2\mathbb{Z})[S^2]; \mathbb{Z}/2\mathbb{Z})\). And so \(\text{red}_*(\mu(M_1 \wedge 1) \wedge \mu(M_1 \wedge 1))) = \xi_1^2 = \text{red}_*(\eta)\).

Recall that we chose our \(\zeta\) so that \(\iota\) would be invertible in \(R \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}[(\zeta)/\zeta^2]\). We know that the invertible elements in \(\mathbb{Z}/2\mathbb{Z}[(\zeta)/\zeta^2]\) are 1 and 1 + \(\zeta\). Both of their squares are equal to 1. So for any \(C \in H_n(R[S^k]; \mathbb{Z}/2\mathbb{Z})\), \([\iota^2 C] = [C]\) as long as \(k > 0\).

Thus, since
\[
\text{red}_*(s(4,\emptyset,\emptyset,\emptyset)) = \text{red}_*(\iota^2 \eta) = \text{red}_*(\eta)
\]
we get by injectivity
\[(3.3.3) \quad [s(4,\emptyset,\emptyset,\emptyset)] = \eta \in H_0(R[S^4]; \mathbb{Z}/2\mathbb{Z}).\]

Now we have found a chain on \(THH^{(k+1)}(R)\) with
\[
\partial(L_1 + L_2 + L + L_h) = \quad 1 \wedge \zeta^k \times \Delta^k + M_* \left( \left( (1 \wedge \zeta^{k-4}) \times \Delta^{k-4} \right) \times \left( (1 \wedge s(4,\emptyset,\emptyset,\emptyset)) \times \Delta^1 \right) \right),
\]

where the first summand in the boundary is of filtration degree \(k\), and the second of filtration degree \(k - 3\). \(1 \wedge \zeta^a \times \Delta^a\) represents \((1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta)\) in the \(E^3\)-term for any \(a\). And by \((3.3.3)\), \(1 \wedge s(4,\emptyset,\emptyset,\emptyset)) \times \Delta^1\) represents \(e_3 = 1 \otimes \eta\) in the \(E^3\)-term. So we get that
\[
d_3(1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta) = (1 \otimes \zeta \otimes \zeta \otimes \cdots \otimes \zeta) \cdot e_3.
\]

\(\square\)

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(3.4) **Addendum to (1.7).**

The following details from Bökstedt’s calculation are used in the calculations; they are listed and explained in the notation established in the section (3.2). Recall however that Bökstedt’s calculation used Milnor’s original classes ξi, and we are using their images under the canonical anti-automorphism; his calculations work with this new choice, and in fact are simplified (see section 7 of [19] and section 4.2 of [12]). Bökstedt’s map

\[ \lambda : S^1 \times \homol_{j \in \mathbb{Z}}(S^j, R[S^j] \wedge S^m) \rightarrow \THH^{(m)}(R) \]

of [3] sends, in our notation, \( \lambda_\ast(\alpha \times C) = (1 \wedge C) \times \Delta^1 \) for \( \alpha \) the 1-simplex in our standard (one 0-simplex, one non-degenerate 1-simplex) model of \( S^1 \), \( m \) the total cardinality of \( j \), and \( C \) a chain on \( R[S^j] \). In his proof of lemma (1.2) in [4], Bökstedt represents the \( \lambda_i \) by \( \bar{\lambda}_i = \bar{\lambda}_1^i \), and calculates that \( \bar{\lambda}_i = [\lambda_\ast(\alpha \times a_i)] \) where the \( a_i \) are cycles for which \( [a_1] = \xi_1 \in H_{m+1}(\mathbb{Z}/2\mathbb{Z}[S^j]; \mathbb{Z}/2\mathbb{Z}) \), and \( [a_i] = (\text{unit}) \cdot \xi_i + (\text{decomposables}) \) for \( i > 1 \). By section 4.2 of [12], with our choice of \( \xi_i \) we get \( [a_i] = \xi_i \) for all \( i \). So if we pick \( \bar{e}_4 = [(1 \wedge \xi_2) \times \Delta^1] \) for some cycle \( \bar{\xi}_2 \) representing \( \xi_2 \in H_{m+2}(\mathbb{Z}[S^j]; \mathbb{Z}/2\mathbb{Z}) \), this clearly represents \( e_4 \) in the \( E_\infty \) term of the spectral sequence calculating the stable homology of \( \THH(Z) \), and the reduction modulo 2 will send \( \bar{e}_4 \rightarrow \bar{\lambda}_2 = \bar{\lambda}_1^2 \). Once \( \bar{e}_4 \) is thus picked, one defines \( \bar{e}_3 = \beta_1(\bar{e}_4) \), which gives \( \bar{e}_3 = [(1 \wedge \eta) \times \Delta^1] \) for a cycle \( \eta \) representing \( \eta \). We know that \( \beta_1(\xi_2) = \eta \) for example by looking at their duals \( \text{Sq}^3 u \) and \( \text{Sq}^2 u \) (where \( u \) is the fundamental class) and recalling that \( \text{Sq}^1 \text{Sq}^2 = \text{Sq}^3 \). This means that \( \bar{e}_3 \) represents the \( e_3 \). To show \( \bar{e}_4 \)’s algebraic independence over \( \mathcal{A} \), Bökstedt uses the fact that it reduces to \( \bar{\lambda}_1^2 \), where \( \bar{\lambda}_1 \) is algebraically independent of \( \mathcal{A} \), and so in particular of \( \overline{\mathcal{A}} \subset \mathcal{A} \).

(3.5) **Proof of Lemma (1.9).** We represent \( \epsilon \) by the homology class \( \bar{\epsilon} \) of the 1-dimensional chain \( 1 \wedge \zeta \times \Delta^1 \) on \( \THH(0) \). This is already a cycle in the unfiltered complex—moreover, it is an absolute cycle so all Bockstein operators vanish on it. Finally, since our representative of \( \bar{\epsilon} \) comes from a chain on 0-dimensional spheres, the multiplication of (1.3.3) gives

\[ M_\ast(\bar{\epsilon} \times \bar{\epsilon}) = 2 \cdot [1 \wedge \zeta \wedge \zeta \times \Delta^2] = 0. \]

\( \square \)

(3.6) **Proof of Lemma (1.10).** We represent \( a_2 \) by the homology class \( \bar{a}_2 \) of the 4-dimensional chain \( (1 \wedge \zeta \wedge \zeta) \times \Delta^2 + (1 \wedge \mu M_1 \wedge \eta) \times \Delta^1 \) on \( \THH(2)(R) \), for \( M_1 \) of (3.3.1). Since

\[ d((1 \wedge \zeta \wedge \zeta) \times \Delta^2) = (1 \wedge 2\zeta) \times \Delta^1 = b((1 \wedge \mu M_1 \wedge \eta) \times \Delta^1) \]

and \( d((1 \wedge \mu M_1 \wedge \eta) \times \Delta^1) = 0 \), this is a cycle in the unfiltered complex. By considering the first summand in the representant we have chosen for \( \bar{a}_2 \), we see that it represents the class \( a_2 \) in the filtered complex. Considering \( (1 \wedge \zeta \wedge \zeta) \times \Delta^2 + (1 \wedge \mu M_1 \wedge \eta) \times \Delta^1 \) as a chain with coefficients in \( \mathbb{Z}/4\mathbb{Z} \), we see that its total boundary

\[ \partial((1 \wedge \zeta \wedge \zeta) \times \Delta^2 + (1 \wedge \mu M_1 \wedge \eta) \times \Delta^1) \]

\[ = 2 \cdot (\zeta \wedge \zeta \times \Delta^1) - 1 \wedge 2\zeta \times \Delta^1 + 1 \wedge 2\zeta \times \Delta^1 - 2 \cdot (1 \wedge \zeta \times \Delta^1) \]

\[ = 2 \cdot (\zeta \wedge \zeta \times \Delta^1 - 1 \wedge \zeta \times \Delta^1) \]

and thus \( \beta_1(\bar{a}_2) = [\zeta \wedge \zeta \times \Delta^1 - 1 \wedge \zeta \times \Delta^1] \), a 3-dimensional homology class on \( \THH(2) \) which is the sum of double suspensions—of the 1-dimensional class \( [\zeta \wedge \zeta \times \Delta^1] \) which we now call
\[ \zeta \epsilon \] (once in each coordinate), and of the 1-dimensional class \([1 \wedge \iota \times \Delta^1]\) (twice in the second coordinate). To analyze the second term, recall from (1.2) that \(\zeta^2 = 2\iota = -2(a\zeta + b)\), or in other words \(\iota = -a\zeta - b\). Thus \([1 \wedge \iota \times \Delta^1] = a[1 \wedge \zeta \times \Delta^1] + b[1 \wedge 1 \times \Delta^1]\) but \(1 \wedge 1 \times \Delta^1 = \partial(1 \wedge 1 \times 1 \times \Delta^2)\), so we are left with \(\beta_1(a_2) = (\zeta + a)\epsilon\). Since we use our names for the designated classes and their suspensions, this shows that \(\beta_1\) applied to an appropriately stabilized \(\tilde{a}_2\) gives an appropriately stabilized \((\zeta + a)\epsilon\).

In order to calculate \(\tilde{a}_2^2\), we compare the \(E^\infty\) terms of Bökstedt's spectral sequences for \(\text{THH}(R)\) and \(\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z})\). Reduction modulo 2 induces a map (see (1.5.1) for the former, and lemma (6.3) in [14] along with proposition (1.15) in [13] for the latter)

\[
\overline{A} \otimes R[\epsilon, a_2, e_4, e_8, \ldots]/(\epsilon^2, a_2^3, \epsilon_2^3 - \text{red} A[\lambda_1, \lambda_2, \lambda_3, \ldots]) \otimes (R \otimes \mathbb{Z}/2\mathbb{Z})[\epsilon]/(\epsilon^2) \otimes \Gamma(\tilde{a}_2).
\]

Bökstedt’s work in [4] shows that \(\text{red}\) embeds \(\overline{A} \hookrightarrow A\) and sends \(e_2 \mapsto \lambda_i\). Clearly the \(R \otimes \mathbb{Z}/2\mathbb{Z}\) in the \((0,0)\) coordinate goes to itself by the identity, and the filtered classes \(e, a_2\) to the classes of the same name in the right hand side. Thus \(\text{red}\) induces an injection on the \(E^\infty\) terms, and so it must induce an injection

\[
H^S_*(\text{THH}(R); \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^S_*(\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}).
\]

The latter is known from [14] to be

\[
H^S_*(\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z}) \cong \Lambda[\lambda_1] \otimes (R \otimes \mathbb{Z}/2\mathbb{Z})[\epsilon]/(\epsilon^2) \otimes \Gamma(\tilde{a}_2).
\]

(Recall that proposition (1.15) in [13] gives the correct multiplication on \(H^S_*(\text{THH}(R \otimes \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})\) since the calculation in [14] uses sums of simplices of the form \(r_0 \wedge r_1 \wedge \cdots \wedge r_k \times \Delta^k\), with \(r_i\) 0-chains on \(R[S^0]\)’s, so \(M\) simply induces a shuffle product on them).

The representative \(\tilde{a}_2\) used in [14] is homologous to \(1 \wedge \zeta \wedge \zeta \times \Delta^3\) (actually it is the sum of this ‘monomial’ with \(\partial((1 \wedge \zeta \times 1 \wedge \zeta) \times \Delta^3))\). From the discussion in (3.4), we know that \(\text{red}_*(((1 \wedge \mu(M_1 \wedge 1)) \times \Delta^1)\) represents the class \(\lambda_1\). We, however, have, a copy of \(1 \wedge \mu(M_1 \wedge \xi) \times \Delta^1\), and while we know that the reduction of \(\iota\) is invertible in \(R \otimes \mathbb{Z}/2\mathbb{Z}\), it may not be equal to 1. We write \(\tilde{\xi}_1 = \text{red}_*\mu(M_1 \wedge 1)\), and note that

\[
\partial(1 \wedge \iota \wedge \tilde{\xi}_1 \times \Delta^2) = (\iota \wedge \tilde{\xi}_1 + 1 \wedge \mu(\iota \wedge \tilde{\xi}_1) + \tilde{\xi}_1 \wedge \iota) \times \Delta^1.
\]

This gives us

\[
\text{red}_*(\tilde{a}_2) = \tilde{a}_2 + \iota \tilde{\lambda}_1 + [(\tilde{\xi}_1 \wedge \iota) \times \Delta^1],
\]

\[
\text{red}_*(\tilde{a}_2^2) \equiv \tilde{a}_2^2 + \iota^2 \tilde{\lambda}_1^2 + [(\tilde{\xi}_1 \wedge \iota) \times \Delta^1]^2 \equiv \iota^2 \tilde{\lambda}_1^2 \equiv \tilde{\lambda}_1^2,
\]

the equality next to last being true since \((\tilde{\xi}_1 \wedge \iota) \times \Delta^1 = M_*((\tilde{\xi}_1 \wedge (1 \wedge \iota \times \Delta^1))\) and \(M_*((1 \wedge \iota \times \Delta^1) \times (1 \wedge \iota \times \Delta^1)) = 2 \cdot (1 \wedge \iota \wedge \iota \times \Delta^2) \equiv 0\), and the last equality being true since the two invertible elements in \(R \otimes \mathbb{Z}/2\mathbb{Z}\) both give 1 when squared. So \(\text{red}_*(\tilde{a}_2^2) \equiv \tilde{\lambda}_1^2\), whereas we know from (3.4) that if we pick \(\tilde{\epsilon}_4 = 1 \wedge \tilde{\xi}_2 \times \Delta^1\) for some chain \(\tilde{\xi}_2\) representing \(\xi_2 \in H_{k+3}(\mathbb{Z}[S^k]; \mathbb{Z}/2\mathbb{Z}) \subset H_{k+3}(R[S^k]; \mathbb{Z}/2\mathbb{Z})\) we will have

\[
\text{red}_*(\tilde{\epsilon}_4) = \tilde{\lambda}_1^2.
\]

Since we know the reduction modulo 2 induces an inclusion on stable homology, this shows that \(\tilde{a}_2^2 = \tilde{\epsilon}_4\). \(\square\)
Proof of Lemma (1.12). Proposition 1.5 in [16] tells us that $\beta_1((\bar{a}_2)_2)$ vanishes. Moreover, part (vi) of this proposition tells us that for an even-dimensional homology class $y$ with mod 2 coefficients on an infinite loop space,

$$\beta_2(y^2) = y\beta_1(y) + Q^\dim y(\beta_1(y)).$$

We want to apply this to $y = \bar{a}_2$. The desired result would follow if we knew that $Q^2(\bar{e}) = 0$, but that follows from Lemma 2.9 in [4], which specializes in our case to the claim that

$$Q^2([1 \smile \zeta \times \Delta^1]) = 1 \smile Q^2(\zeta) \times \Delta^1 = 1 \smile \zeta Q^2(1) = 0$$

where the equality before last follows from the fact that the multiplication of $R$'s Eilenberg-MacLane spectrum commutes with multiplication by $R$, and the last equality follows from the fact that $Q^0$ is the only $Q$-operation which does not vanish on 1 (see for example part (4) of theorem 1.1 in chapter III of [8]). For a multiple of $\bar{e}$ by an element $r \in R$, we observe that in $\text{THH}(R)$, too, all the structure we have defined commutes with multiplication by elements of $R$, and thus $Q^2(r\bar{e}) = r^2Q^2(\bar{e}) = 0$. \hfill \Box

REFERENCES

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