PROPERLY ERGODIC STRUCTURES

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Abstract. We study ergodic $\text{Sym}(\mathbb{N})$-invariant probability measures on the space of $L$-structures with domain $\mathbb{N}$. We call such measures “ergodic structures”. In particular, we are interested in the properly ergodic case, in which no isomorphism class has measure 1. A Morley–Scott analysis shows that proper ergodicity can always be explained by a splitting of measure over continuum-many types in a countable fragment of $\mathcal{L}_{\omega_1, \omega}$. This implies that the theory of the ergodic structure in any countable fragment of $\mathcal{L}_{\omega_1, \omega}$ has continuum-many models up to isomorphism, an analogue of Vaught’s Conjecture in this context. We use the Aldous–Hoover–Kallenberg theorem to show that a structure sampled from a properly ergodic source almost surely satisfies a condition we call “rootedness” on the realizations of the types of measure 0. Finally, we show that a single rooted model of a theory $T$ with trivial definable closure can be used to construct continuum-many distinct properly ergodic structures which almost surely satisfy $T$. As a consequence, we obtain a characterization of those theories in countable fragments of $\mathcal{L}_{\omega_1, \omega}$ which admit properly ergodic models.

1. Introduction

In this paper, we study random countable structures in a countable relational language. A basic example is the Erdős–Rényi random graph with domain
\( \mathbb{N} \) obtained by independently flipping a coin for each potential edge \( \{i, j\} \). This process produces the so-called Rado graph, up to isomorphism, with probability 1. (The Rado graph is also called the random graph, because of this construction.) As another motivating class of examples, given a Borel graph \( G \) and a probability measure \( \nu \) on \( G \), consider the random induced subgraph given by sampling countably many points i.i.d. according to \( \nu \). More examples are given in Section 3.

We write \( \text{Str}_L \) for the measurable space of \( L \)-structures with domain \( \mathbb{N} \). We say that a probability measure on \( \text{Str}_L \) is \textit{invariant} when it is invariant under the natural action (called the \textit{logic action}) of the permutation group \( \text{Sym}(\mathbb{N}) \) on \( \text{Str}_L \). This captures the idea that a random structure has a distribution that is isomorphism-invariant, or insensitive to the labeling of the domain. An invariant probability measure \( \mu \) on \( \text{Str}_L \) is \textit{ergodic} when the null and co-null sets are the only Borel sets which are almost surely invariant. That is, if \( \mu(X \triangle \sigma[X]) = 0 \) for all \( \sigma \in \text{Sym}(\mathbb{N}) \), then \( \mu(X) = 0 \) or 1.

We are interested in ergodic invariant probability measures on \( \text{Str}_L \). We call such measures \textit{ergodic structures}, as one may view the ergodic invariant measures as random analogues of model-theoretic structures. There are several reasons to focus on ergodic structures:

1. An ergodic structure \( \mu \) determines the “almost-sure truth value” of every sentence of the infinitary logic \( L_{\omega_1, \omega} \). Indeed, the set of models for a sentence of \( L_{\omega_1, \omega} \) is an invariant Borel set in \( \text{Str}_L \), and hence is assigned measure 0 or 1 by \( \mu \). This allows us to define the complete almost-sure theory of \( \mu \), in \( L_{\omega_1, \omega} \) or any fragment of \( L_{\omega_1, \omega} \).

2. The ergodic structures are extreme points in the space of invariant probability measures on \( \text{Str}_L \), and any invariant probability measure can be decomposed as a mixture of ergodic ones. (For details, see, e.g., [Kal05, Lemma A1.2 and Theorem A1.3].)

3. The Aldous–Hoover–Kallenberg theorem [Ald81], [Hoo79], [Kal92] implies that every invariant probability measure on \( \text{Str}_L \) can be represented as a random process that depends on independent sources of randomness at every finite subset of \( \mathbb{N} \) (see Section 2.4 for more details). The ergodic structures are those invariant measures with dissociated representations, in which the random process does not depend on randomness at the level of the empty set, or, equivalently, in which the behavior on disjoint finite subsets of \( \mathbb{N} \) is independent.

The Aldous–Hoover–Kallenberg theorem represents an ergodic invariant measure on the space of graphs with domain \( \mathbb{N} \) in essentially the same way as a graphon [Lov12]; if we drop the ergodic condition, we get a mixture of graphons. Just as graphons arise as limits of sequences of finite graphs which are convergent in the appropriate sense, random structures in our sense can be viewed as limits of convergent sequences
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of finite $L$-structures. See the survey [Aus08] for more information on this perspective.

Many natural examples of ergodic structures, such as the probabilistic construction of the Rado graph described above, are almost surely isomorphic to a single countable $L$-structure; such an ergodic structure concentrates on a single $\text{Sym}(\mathbb{N})$-orbit. In contrast to this behavior, we call an ergodic structure properly ergodic when it does not assign measure 1 to any $\text{Sym}(\mathbb{N})$-orbit. A properly ergodic structure assigns measure 0 to every isomorphism class of $L$-structures.

The paper [AFP16a] characterized those countable structures $M$ for which there exists an invariant measure concentrating on (the isomorphism class of) $M$; in fact, an ergodic structure was constructed concentrating on each such structure’s isomorphism class. The key property in the characterization is trivial definable closure (see Section 2.2). Later this was extended in [AFP16b] to a characterization of those theories (in any countable fragment $F$ of $\mathcal{L}_{\omega_1,\omega}$) which appear as the complete $F$-theory $\text{Th}_F(\mu)$ of an ergodic structure $\mu$. The case of proper ergodic structures was taken up in [AFNP16], where a class of examples was constructed, concentrating on the sets of models of certain “approximately $\aleph_0$-categorical” first-order theories with trivial definable closure.

In the present paper, we characterize, for countable fragments $F$ of $\mathcal{L}_{\omega_1,\omega}$, the complete $F$-theories of properly ergodic structures. Additionally, we show that for any properly ergodic structure $\mu$, the complete $\mathcal{L}_{\omega_1,\omega}$-theory $\text{Th}(\mu)$ has no models (of any cardinality), but that for any countable fragment $F$, the $F$-theory $\text{Th}_F(\mu)$ has continuum-many models up to isomorphism. This can be viewed as an analogue of Vaught’s Conjecture.

In [AFKP15], it was shown that for every countable structure $M$ with trivial definable closure, the number of distinct ergodic structures concentrating on the isomorphism class of $M$ is either one or continuum. Moreover, the case of one only occurs when $M$ is highly homogeneous, i.e. interdefinable with one of the five reducts of the dense linear order. We also extend this result to the properly ergodic case, showing that if there exists a properly ergodic structure concentrating on the models of an $F$-theory $T$, then there are continuum-many distinct properly ergodic structures concentrating on the models of $T$.

Section 2 contains the basic definitions and background theory. In Section 3, we provide a number of examples of properly ergodic structures, which illustrate some of their key features.

Our results begin in Section 4, where we undertake a Morley–Scott analysis of an ergodic structure $\mu$ (based on Morley’s proof [Mor70] that the number of isomorphism classes of countable models of a sentence of $\mathcal{L}_{\omega_1,\omega}$ is countable, $\aleph_1$, or $2^{\aleph_0}$). This gives us a notion of Scott rank for ergodic structures and, in the properly ergodic case, allows us to find a countable fragment $F$ of $\mathcal{L}_{\omega_1,\omega}$ in which there is a formula $\chi(\bar{x})$ which has positive measure, but which picks
out continuum-many $F$-types, each of which has measure 0. The analogue of Vaught’s Conjecture mentioned above is a corollary of this analysis.

In Section 5, we introduce the notion of a rooted model of a theory. A model $M$ is rooted if a collection of non-isolated types (e.g., the continuum-many types of measure 0 coming from the Morley–Scott analysis) has few realizations in $M$ in a precise sense. We use the Aldous–Hoover–Kallenberg theorem to show that a model sampled from a properly ergodic measure is almost surely rooted.

In Section 6, we use a single rooted model of a theory $T$ with trivial definable closure to guide the construction, via an inverse limit, of a rooted Borel model $M \models T$, equipped with an atomless probability measure $\nu$. Then a properly ergodic structure $\mu$ concentrating on the models of $T$ is obtained by sampling from $(M, \nu)$. The inverse limit construction is a refinement of the methods from [AFP16a], [AFP16b], and [AFNP16], which in turn generalized a construction due to Petrov and Vershik [PV10]. Further, we use a technique from [AFKP15] to rescale $\nu$, obtaining continuum-many distinct properly ergodic structures concentrating on the models of $T$.

Putting together the results of Sections 4–6, we obtain the characterization of the complete $F$-theories of properly ergodic structures.

2. Preliminaries

2.1. The space $\text{Str}_L$, infinitary logic, and ergodic structures. Throughout this paper, let $L$ be a countable relational language. We study invariant measures on the space $\text{Str}_L$ of $L$-structures with domain $\mathbb{N}$. One could formulate this work in terms of arbitrary countable languages (which allow constant and function symbols), but it turns out that one does not lose much by working in the relational case — there are no ergodic structures in languages having constant symbols, and, in an ergodic structure, the interpretation of a function symbol must take some value among its inputs almost surely (see [AFP16a, §§3–4]). For more details on how to translate results about invariant measures for countable relational languages to the case of arbitrary countable languages, see [AFP16b].

**Definition 2.1.** $\text{Str}_L$ is the space of $L$-structures with domain $\mathbb{N}$. The topology is generated by the sets of the form $[R(\bar{a})] = \{M \in \text{Str}_L \mid M \models R(\bar{a})\}$ and $[\neg R(\bar{a})] = \{M \in \text{Str}_L \mid M \models \neg R(\bar{a})\}$, where $R$ ranges over the relation symbols in $L$ and $\bar{a}$ ranges over the $\text{ar}(R)$-tuples from $\mathbb{N}$.

A structure $M \in \text{Str}_L$ is uniquely determined by whether or not, for each relation symbol $R$ in $L$ of arity $\text{ar}(R)$ and each $\text{ar}(R)$-tuple $\bar{a}$ from $\mathbb{N}$,

$$M \models R(\bar{a})$$

holds. It follows that $\text{Str}_L$ is homeomorphic to the Cantor space

$$\prod_{R \in L} 2^{(\text{ar}(R))}.$$
Recall that $\mathcal{L}_{\omega_1,\omega}$ is the infinitary extension of first-order logic obtained by allowing, as new formula-building operations, the conjunction or disjunction of any countable ($< \omega_1$) family of formulas with a common finite ($< \omega$) set of free variables. We ensure that all our variables come from a fixed countable supply. For a reference on $\mathcal{L}_{\omega_1,\omega}$, see [KK04].

Given a formula $\varphi(\bar{x}) \in \mathcal{L}_{\omega_1,\omega}$ and a tuple $\bar{a}$ from $\mathbb{N}$ of the same length as $\bar{x}$, we let

$$J_{\varphi}(\bar{a}) = \{ M \in \text{Str}_L \mid M \models \varphi(\bar{a}) \}.$$  

Every $J_{\varphi}(\bar{a})$ is a Borel set in $\text{Str}_L$, and when $\varphi(\bar{x})$ is a quantifier-free first-order formula, $J_{\varphi}(\bar{a})$ is clopen. Indeed, negation and (finite/countable) conjunction and disjunction correspond to (finite/countable) Boolean operations, and quantifiers over the countable domain also correspond to countable Boolean operations:

$$\forall x \varphi(\bar{a}, x) = \bigcap_{b \in \mathbb{N}} [\varphi(\bar{a}, b)]$$
$$\exists x \varphi(\bar{a}, x) = \bigcup_{b \in \mathbb{N}} [\varphi(\bar{a}, b)].$$

In fact, by compactness, every clopen set in $\text{Str}_L$ has the form $J_{\varphi}(\bar{a})$ for some quantifier-free first-order formula $\varphi$.

Let $\text{Sym}(\mathbb{N})$ denote the permutation group of $\mathbb{N}$.

**Definition 2.2.** The **logic action** is the natural action of $\text{Sym}(\mathbb{N})$ on $\text{Str}_L$, given by permuting the underlying set. Explicitly, for $\sigma \in \text{Sym}(\mathbb{N})$ and $M \in \text{Str}_L$, we have

$$\sigma(M) \models R(a_1, \ldots, a_n) \quad \text{if and only if} \quad M \models R(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)).$$

Note that $\sigma(M) = N$ if and only if $\sigma: M \to N$ is an isomorphism, so the orbit of a point $M \in \text{Str}_L$ under the logic action is the set of all structures in $\text{Str}_L$ which are isomorphic to $M$. We recall Scott’s theorem, which says that this set is definable by a sentence of $\mathcal{L}_{\omega_1,\omega}$.

**Theorem 2.3** (Scott, [Mar02, Theorem 2.4.15]). For any countable structure $M$, there is a sentence $\varphi_M$ of $\mathcal{L}_{\omega_1,\omega}$, the **Scott sentence of** $M$, such that for all countable structures $N$, we have $N \models \varphi_M$ if and only if $N \cong M$.

We are now able to define the class of invariant measures on $\text{Str}_L$, and specifically, the ergodic and properly ergodic ones.

**Definition 2.4.** Let $\mu$ be a Borel probability measure on $\text{Str}_L$. We say that $\mu$ is **invariant** (under the logic action) if, for every Borel set $X$ and every $\sigma \in \text{Sym}(\mathbb{N})$, we have $\mu(\sigma[X]) = \mu(X)$.

Now suppose that $\mu$ is invariant. A Borel set $X$ is **almost surely invariant** if $\mu(X \triangle \sigma[X]) = 0$ for all $\sigma \in \text{Sym}(\mathbb{N})$. We say that $\mu$ is **ergodic** if, for every almost surely invariant Borel set $X$, either $\mu(X) = 0$ or $\mu(X) = 1$. Following
terminology from [BM00, §I.2] and elsewhere, we say that \( \mu \) is properly ergodic if \( \mu(X) = 0 \) for every orbit \( X \) of the logic action.

**Definition 2.5.** An ergodic structure is an ergodic invariant probability measure on \( \text{Str}_L \).

This definition takes on a more concrete character if we restrict our attention to the measures assigned to instances of quantifier-free formulas. The following proposition is an application of the Hahn–Kolmogorov measure extension theorem [Tao11, Theorem 1.7.8, Exercise 1.7.7].

**Proposition 2.6.** Let \( B^* \) be the Boolean algebra of clopen sets in \( \text{Str}_L \) (so \( B^* \) consists of those sets of the form \( [\varphi(\bar{a})] \), where \( \varphi \) is a quantifier-free first-order formula and \( \bar{a} \) is a tuple from \( N \)). Any finitely additive measure \( \mu^* \) on \( B^* \) extends to a unique Borel probably measure \( \mu \) on \( \text{Str}_L \). Moreover, \( \mu \) is invariant if and only if \( \mu^* \) is; that is, if and only if \( \mu^*([\varphi(\bar{a})]) = \mu^*([\varphi(\sigma(\bar{a}))]) \) for any \( \sigma \in \text{Sym}(N) \).

**Remark 2.7.** Additionally, it follows from Theorem 2.26 below that an invariant measure \( \mu \) on \( \text{Str}_L \) is ergodic if and only if the quantifier-free types of disjoint tuples from \( N \) are independent. That is, whenever \( \varphi(\bar{x}) \) and \( \psi(\bar{y}) \) are quantifier-free first-order formulas and \( \bar{a} \) and \( \bar{b} \) are disjoint tuples from \( N \), we have \( \mu([\varphi(\bar{a}) \land \psi(\bar{b})]) = \mu([\varphi(\bar{a})]) \mu([\varphi(\bar{b})]) \).

For the remainder of this section, let \( \mu \) be an ergodic structure.

If \( \varphi(\bar{x}) \) is a formula of \( L_{\omega_1,\omega} \) and \( \bar{a} \) is a tuple of distinct elements of \( N \), then, since \( \mu \) is invariant for the logic action, the measure \( \mu([\varphi(\bar{a})]) \) is independent of the choice of \( \bar{a} \). For convenience, we denote this quantity by \( \mu(\varphi(\bar{a})) \). Note that under this convention, if \( \varphi(\bar{x}) \) implies \( x_i = x_j \) for some \( i \neq j \), then \( \mu(\varphi(\bar{x})) = 0 \).

**Definition 2.8.** If \( \varphi \) is a sentence of \( L_{\omega_1,\omega} \), we say \( \mu \) almost surely satisfies \( \varphi \), or \( \mu \) concentrates on \( \varphi \), if \( \mu(\varphi) = 1 \). We write \( \mu \models \varphi \), and we set

\[
\text{Th}(\mu) = \{ \varphi \in L_{\omega_1,\omega} \mid \mu \models \varphi \}.
\]

Similarly, if \( \Sigma \) is a set of sentences of \( L_{\omega_1,\omega} \), we write \( \mu \models \Sigma \) if \( \mu \models \varphi \) for all \( \varphi \in \Sigma \).

The following result is a connection between infinitary logic and ergodic invariant measures; see also [AFP16b].

**Proposition 2.9.** \( \text{Th}(\mu) \) is a complete and countably consistent theory of \( L_{\omega_1,\omega} \). That is, for every sentence \( \varphi \) of \( L_{\omega_1,\omega} \), \( \varphi \in \text{Th}(\mu) \) or \( \neg \varphi \in \text{Th}(\mu) \), and every countable subset \( \Sigma \subseteq \text{Th}(\mu) \) has a model.

**Proof.** For any sentence \( \varphi \), the set \( [\varphi] \) is an invariant Borel set. In particular, it is almost surely invariant, so by ergodicity, \( \mu(\varphi) = 0 \) or \( 1 \), and hence \( \mu \models \varphi \) or \( \mu \models \neg \varphi \). Now let \( \Sigma \) be a countable subset of \( \text{Th}(\mu) \). Since a countable
intersection of measure 1 sets has measure 1, \( \mu(\bigwedge_{\varphi \in \Sigma} \varphi) = 1 \). In particular, \( \bigwedge_{\varphi \in \Sigma} \varphi \) is non-empty.

A special case of Definition 2.8 is when the sentence \( \varphi \) is a Scott sentence.

**Definition 2.10.** If \( M \) is a countable structure, we say that \( \mu \) is **almost surely isomorphic to** \( M \), or \( \mu \) **concentrates on** \( M \), if \( \mu \mid_{\varphi} = \varphi_M \), where \( \varphi_M \) is the Scott sentence of \( M \); equivalently, \( \mu \) assigns measure 1 to the orbit of \( M \).

**Remark 2.11.** If \( \mu \) is properly ergodic, then \( \text{Th}(\mu) \) contains \( \neg \varphi_M \) for every countable structure \( M \), and thus \( \text{Th}(\mu) \) has no countable models. A priori, \( \text{Th}(\mu) \) may have uncountable models (Łoś–Skolem does not apply to complete theories of \( \mathcal{L}_{\omega_1,\omega} \)), but we will see later (Corollary 4.9) that this is not the case: \( \text{Th}(\mu) \) has no models of any cardinality. Nevertheless, as noted in Proposition 2.9, every countable subset of \( \text{Th}(\mu) \) has countable models. This suggests that we should restrict our attention to countable fragments of \( \mathcal{L}_{\omega_1,\omega} \).

**Definition 2.12.** A **fragment** of \( \mathcal{L}_{\omega_1,\omega} \) is a set of formulas which contains all atomic formulas and is closed under subformulas, finite Boolean combinations, quantification, and substitution of free variables (from the countable supply). If \( F \) is a fragment of \( \mathcal{L}_{\omega_1,\omega} \), we set

\[
\text{Th}_F(\mu) = \{ \varphi \in F \mid \mu \models \varphi \}.
\]

A countable set of formulas \( \Phi \) generates a countable fragment \( \langle \Phi \rangle \), the least fragment containing this set. The minimal fragment \( \text{FO} := \langle \emptyset \rangle \) is first-order logic.

**Definition 2.13.** Let \( F \) be a countable fragment of \( \mathcal{L}_{\omega_1,\omega} \).

- A set of sentences \( T \) is a (complete satisfiable) **\( F \)-theory** if \( T \) has a model and, for every sentence \( \varphi \in F \), either \( \varphi \in T \) or \( \neg \varphi \in F \). Equivalently, \( T = \{ \psi \in F \mid M \models \psi \} \) for some structure \( M \).
- A set of formulas \( p(\bar{x}) \) is an **\( F \)-type** if there is a structure \( M \) and a tuple \( \bar{a} \) from \( M \) such that \( p(\bar{x}) = \{ \psi(\bar{a}) \in F \mid M \models \psi(\bar{a}) \} \). We say that \( \bar{a} \) **realizes** \( p \) in \( M \).
- An \( F \)-type \( p \) is **consistent** with an \( F \)-theory \( T \) if it is realized in some model of \( T \), and we write \( S^n_F(T) \) for the set of \( F \)-types in \( n \) variables which are consistent with \( T \).

**Remark 2.14.** The Łoś–Skolem theorem holds for countable fragments of \( \mathcal{L}_{\omega_1,\omega} \) (see [KK04, Theorem 1.5.4]). Thus, if \( F \) is countable, every \( F \)-theory has a countable model and every \( F \)-type which is consistent with \( T \) is realized in a countable model of \( T \).

If \( F \) is countable and \( p \) is an \( F \)-type, then we denote by \( \theta_p(\bar{x}) \) the conjunction of all the formulas in \( p \), \( \bigwedge_{\varphi \in p} \varphi(\bar{x}) \). This is a formula of \( \mathcal{L}_{\omega_1,\omega} \) (although not a formula of \( F \) in general), so it is assigned a measure by our ergodic structure \( \mu \).
We will write $\mu(p)$ as shorthand for $\mu(\theta_p(\pi))$. This is the probability, according to $\mu$, that any given tuple of distinct elements of $\mathbb{N}$ satisfies $p$.

2.2. Trivial definable closure. The paper [AFP16b] shows that the characteristic property satisfied by the theory (in a fragment) of an ergodic structure is trivial definable closure. Here we state several definitions and basic facts, and provide a proof of one direction of this characterization.

Definition 2.15. Let $F$ be a fragment of $L_{\omega_1\omega}$. An $F$-theory $T$ has trivial definable closure (abbreviated trivial dcl) if there is no formula $\varphi(\pi, y)$ in $F$ such that

$$T \models \exists \pi \exists! y \left( (\bigwedge_{i=1}^n y \neq x_i) \land \varphi(\pi, y) \right).$$

Here $\exists! y$ is the standard abbreviation for “there exists a unique $y$”.

Remark 2.16. If $T$ is the complete $F$-theory of a structure $M$, then $T$ has trivial dcl if and only if $M$ has trivial dcl for the fragment $F$ in the usual sense: $\text{dcl}_F(A) = A$ for all $A \subseteq M$, where $\text{dcl}_F(A)$ is the set of all $b \in M$ such that $b$ is the unique element of $M$ satisfying some formula in $F$ with parameters from $A$.

If $\varphi(\pi, y)$ witnesses that $T$ has nontrivial dcl, then taking $\varphi^*$ to be the formula $\varphi(\pi, y) \land \exists^{\leq 1} y \varphi(\pi, y)$ we have the stronger condition that $T$ proves that $\varphi^*$ is a definable function on some non-empty domain. That is,

$$T \models (\exists \pi \exists y \left( \bigwedge_{i=1}^n y \neq x_i \right) \land \varphi^*(\pi, y)) \land \left( \forall \pi \exists^{\leq 1} y \varphi^*(\pi, y) \right).$$

Here $\exists^{\leq 1} y$ is the standard abbreviation for “there is at most one $y$”.

The following argument first appeared (in a slightly different setting) in [AFP16a, Theorem 4.1]; as stated, this result is from [AFP16b]. We include it here for completeness.

The key observation is the standard fact that if a measure is invariant under the action of some group $G$, then no positive-measure set can have infinitely many almost surely disjoint images under the action of $G$.

Theorem 2.17. Let $\mu$ be an ergodic structure and $F$ a fragment of $L_{\omega_1\omega}$. Then $\text{Th}_F(\mu)$ has trivial dcl.

Proof. Suppose there is a formula $\varphi(\pi, y)$ in $F$ such that

$$\mu \left( \exists \pi \exists! y \left( (\bigwedge_{i=1}^n y \neq x_i) \land \varphi(\pi, y) \right) \right) = 1.$$

Let $\psi(\pi, y)$ be the formula $(\bigwedge_{i=1}^n y \neq x_i) \land \varphi(\pi, y)$.

By countable additivity of $\mu$, there is a tuple $\pi$ from $\mathbb{N}$ such that

$$\mu(\exists! y \psi(\pi, y)) > 0.$$

Let $\theta(\pi)$ be the formula $\forall z_1 \forall z_2 (\psi(\pi, z_1) \land \psi(\pi, z_2) \rightarrow (z_1 = z_2))$, so that $\exists! y \psi(\pi, y)$ is equivalent to $\exists y \psi(\pi, y) \land \theta(\pi)$. 


Since this formula has positive measure, countable additivity again implies that there is some \( b \in \mathbb{N} \setminus \bar{a} \) such that
\[
\beta := \mu([\psi(\bar{a}, b) \land \theta(\bar{a})]) > 0.
\]
By invariance, for any \( c \in \mathbb{N} \setminus \bar{a} \), we also have
\[
\mu([\psi(\bar{a}, c) \land \theta(\bar{a})]) = \beta.
\]
But \( \theta \) ensures that \( \psi(\bar{a}, b) \land \theta(\bar{a}) \) and \( \psi(\bar{a}, c) \land \theta(\bar{a}) \) are inconsistent when \( b \neq c \), so, computing the measure of the disjoint union,
\[
\mu \left( \bigcup_{b \in \mathbb{N} \setminus \bar{a}} [\psi(\bar{a}, b) \land \theta(\bar{a})] \right) = \sum_{b \in \mathbb{N} \setminus \bar{a}} \beta = \infty,
\]
which is impossible. \( \square \)

In the language of Section 2.1, the main result of [AFP16a] was a characterization of those countable structures \( M \) such that there exists an ergodic structure \( \mu \) which is almost surely isomorphic to \( M \). That characterization was given in terms of trivial “group-theoretic” dcl (where the group is \( \text{Aut}(M) \)).

**Definition 2.18.** A countable structure \( M \) has **trivial group-theoretic dcl** if for any finite subset \( A \subseteq M \) and element \( b \in M \setminus A \), there is an automorphism \( \sigma \in \text{Aut}(M) \) such that \( \sigma(a) = a \) for all \( a \in A \), but \( \sigma(b) \neq b \).

**Theorem 2.19** ([AFP16a, Theorem 1.1]). Let \( M \) be a countable structure. There exists an ergodic structure concentrating on \( M \) if and only if \( M \) has trivial group-theoretic dcl.

**Remark 2.20.** The method in [AFP16a] of obtaining a measure via i.i.d. sampling from a Borel structure, which we use again in Section 6, always produces an ergodic measure. This was mentioned in passing in [AFP16a], though not stated as part of the main theorem; for a proof, see [AFKP15, Proposition 2.23]. See also Theorem 2.26 and Lemma 6.2 below.

It is a consequence of Scott’s Theorem (Theorem 2.3) that the notion of trivial group-theoretic dcl for a countable structure \( M \) is equivalent to the usual (syntactic) trivial dcl for \( \text{Th}_{F_M}(M) \) in an appropriate countable fragment \( F_M \) of \( \mathcal{L}_{\omega_1, \omega} \). That is, given a finite subset \( A \) of \( M \), an element \( b \in M \) is fixed by all automorphisms fixing \( A \) pointwise if and only if there is a formula from \( F_M \) with parameters from \( A \) which uniquely defines \( b \) in \( M \).

Unlike the group-theoretic notion of trivial dcl, which is defined for a given structure, the syntactic notion of trivial dcl (Definition 2.15) is defined for theories in arbitrary countable fragments, and so is the relevant notion for this paper.

**2.3. Pithy \( \Pi_2 \) theories.** It is a well-known fact, originally due to Chang, that if \( T \) is a theory in a countable fragment \( F \) of \( \mathcal{L}_{\omega_1, \omega} \), then the models of \( T \) are
exactly the reducts to $L$ of the models of a first-order theory $T'$ in a larger language $L' \supseteq L$ that omit a countable set of types $Q$.

The idea is to Morleyize: we introduce a new relation symbol $R\varphi$ for every formula $\varphi(\bar{x})$ in $F$ and encode the intended interpretations of the $R\varphi$ in the theory $T'$. The role of the countable set of types $Q$ is to achieve this for infinitary conjunctions and disjunctions, which cannot be accounted for in first-order logic.

There are two features of this construction that will be useful for us. First, it reduces $F$-types to quantifier-free types. Second, $T'$ can be axiomatized by pithy $\Pi_2$ sentences, also called “one point extension axioms”.

**Definition 2.21.** A first-order sentence is pithy $\Pi_2$ if it is universal ($\Pi_1$) or if it has the form $\forall \bar{x} \exists y \varphi(\bar{x}, y)$, where $\varphi(\bar{x}, y)$ is quantifier-free, $\bar{x}$ is a tuple of variables (possibly empty), and $y$ is a single variable. A pithy $\Pi_2$ theory is a set of pithy $\Pi_2$ sentences.

Note that, in the context of this paper, all pithy $\Pi_2$ theories are first-order.

**Theorem 2.22.** Let $F$ be a countable fragment of $\mathcal{L}_{\omega_1, \omega}$ and $T$ an $F$-theory. Then there is a language $L' \supseteq L$, an $L'$-theory $T'$ that is pithy $\Pi_2$, and a countable set of partial quantifier-free $L'$-types $Q$ such that the following hold.

(a) There is a bijection between formulas $\varphi(\bar{x})$ in $F$ and atomic $L'$-formulas $R\varphi(\bar{x})$ which are not in $L$, such that if $M \models T'$ omits all the types in $Q$, then $M \models \forall \bar{x} \varphi(\bar{x}) \iff R\varphi(\bar{x})$.

(b) The reduct to $L$ is a bijection between the class of models of $T'$ omitting all the types in $Q$ and the class of models of $T$.

**Proof.** Let $L' = L \cup \{R\varphi \mid \varphi(\bar{x}) \in F\}$, where the arity of the relation symbol $R\varphi$ is the length of the tuple $\bar{x}$. By convention, we allow 0-ary relation symbols (i.e., propositional symbols). Thus, we include a 0-ary relation $R\psi$ for every sentence $\psi \in F$.

Let $T_{\text{def}}$ be the theory consisting of the following axioms, for each formula $\varphi(\bar{x}) \in F$:

1. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff \varphi(\bar{x}) \right)$, if $\varphi(\bar{x})$ is atomic.
2. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff \neg R\psi(\bar{x}) \right)$, if $\varphi$ is of the form $\neg \psi(\bar{x})$.
3. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff R\psi(\bar{x}) \land R\theta(\bar{x}) \right)$, if $\varphi$ is of the form $\psi(\bar{x}) \land \theta(\bar{x})$.
4. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff R\psi(\bar{x}) \lor R\theta(\bar{x}) \right)$, if $\varphi$ is of the form $\psi(\bar{x}) \lor \theta(\bar{x})$.
5. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff R\psi_i(\bar{x}) \right)$ for all $i$, if $\varphi$ is of the form $\bigwedge_{i \in I} \psi_i(\bar{x})$.
6. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff R\theta(\bar{x}) \right)$ for all $i$, if $\varphi$ is of the form $\bigvee_{i \in I} \psi_i(\bar{x})$.
7. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff \forall y R\psi(\bar{x}, y) \right)$, if $\varphi$ is of the form $\forall y \psi(\bar{x}, y)$.
8. $\forall \bar{x} \left( R\varphi(\bar{x}) \iff \exists y R\psi(\bar{x}, y) \right)$, if $\varphi$ is of the form $\exists y \psi(\bar{x}, y)$.

Note that all the axioms of $T_{\text{def}}$ are first-order and universal except for those of type (7) and (8), which are pithy $\Pi_2$ when put in prenex normal form.

The axioms of type (5) and (6) cannot be made into bi-implications, since arbitrary countable infinite conjunctions and disjunctions are not expressible
in first-order logic. To ensure that the corresponding \( R_\varphi \) have their intended interpretation, we let \( Q \) consist of the partial quantifier-free types:

(i) \( q_\varphi(x) = \{ R_\psi(x) \mid i \in I \} \cup \{ \neg R_\varphi(x) \} \), for all \( \varphi(x) \) of the form \( \bigwedge_{i \in I} \psi_i(x) \).

(ii) \( q_\varphi(x) = \{ \neg R_\psi(x) \mid i \in I \} \cup \{ R_\varphi(x) \} \), for all \( \varphi(x) \) of the form \( \bigvee_{i \in I} \psi_i(x) \).

It is now straightforward to show by induction on the complexity of formulas that if a model \( M \models T \) omits every type in \( Q \), then for all \( \varphi(x) \) in \( F \) and all \( a \) from \( M \), \( M \models \varphi(a) \) if and only if \( M \models R_\varphi(a) \). This establishes (a). It also implies that every \( L \)-structure \( N \) admits a unique expansion to an \( L' \)-structure \( N' \) which is a model of \( T_{def} \) and omits every type in \( Q \). As a consequence, if we set \( T' = T_{def} \cup \{ R_\psi \mid \psi \in T \} \), then the following hold.

- If \( M \) is a model of \( T' \) which omits every type in \( Q \), then the reduct \( M \restriction L \) is a model of \( T \).
- If \( N \models T \), then \( N' \models T' \), where \( N' \) is the canonical expansion of \( N \).

This establishes (b). \( \square \)

**Corollary 2.23.** There is a bijection between the ergodic \( L \)-structures (ergodic invariant measures on \( \text{Str}_L \)) which almost surely satisfy \( T \) and the ergodic \( L' \)-structures (ergodic invariant measures on \( \text{Str}_{L'} \)) which almost surely satisfy \( T' \) and omit all the types in \( Q \). This bijection sends ergodic structures to ergodic structures and properly ergodic structures to properly ergodic structures.

**Proof.** The reduct \( \models L \) is a continuous map \( \text{Str}_{L'} \to \text{Str}_L \), since the preimages of clopen sets in \( \text{Str}_L \) are also clopen sets in \( \text{Str}_{L'} \). By Theorem 2.22, \( \models L \) is a bijection between the subspace \( X' \) of \( \text{Str}_{L'} \) consisting of models of \( T' \) which omit all the types in \( Q \) and the subspace \( X \) of \( \text{Str}_L \) consisting of models of \( T \). Upon restricting to these subspaces, the inverse of \( \models L \) is a Borel map, since the image of a clopen set in \( X' \) (described by a first-order quantifier-free formula) is a Borel set in \( X \) (described by a formula of \( L_{\omega_1 \omega} \)). Hence \( \models L \) is a Borel isomorphism between these subspaces, and it induces a bijection between the set of probability measures on \( \text{Str}_{L'} \) concentrating on \( X' \) and the set of probability measures on \( \text{Str}_L \) concentrating on \( X \). Moreover, \( \models L \) preserves the logic action, so the induced bijection on measures preserves invariance, ergodicity, and proper ergodicity. \( \square \)

### 2.4. The Aldous–Hoover–Kallenberg theorem and representations.

In this section, we state a version of the Aldous–Hoover–Kallenberg theorem. This theorem, which is a generalization of de Finetti’s theorem to exchangeable arrays of random variables, was discovered independently by Aldous [Ald81] and Hoover [Hoo79], and further developed by Kallenberg [Kal92] and others. For proofs, we direct the reader to Kallenberg’s book [Kal05, Chapter 7]. See [Ack15, Section 2.5] for a discussion of how to translate from the purely probabilistic statements in Kallenberg to the setting here, involving spaces of quantifier-free types. The survey by Austin [Aus08] provides details on its application to random structures.
We denote by \([n]\) the set \(\{0, \ldots, n-1\}\), by \(\mathbb{N}^{[n]}\) the set of \(n\)-tuples of distinct elements of \(\mathbb{N}\) (that is, injective functions \([n] \rightarrow \mathbb{N}\)), and by \(\mathcal{P}_\text{fin}(\mathbb{N})\) the set of all finite subsets of \(\mathbb{N}\). Given a tuple \(\overline{a} \in \mathbb{N}^{[n]}\), we denote by \(\|\overline{a}\|\) the set in \(\mathcal{P}_\text{fin}(\mathbb{N})\) enumerated by \(\overline{a}\).

Let \(S^{\text{qf}}_n(L)\) be the Stone space of quantifier-free \(n\)-types. Its points are the complete first-order quantifier-free types in the variables \(x_0, \ldots, x_{n-1}\), and its topology is generated by the clopen sets \(\llbracket \varphi(\overline{x}) \rrbracket = \{p(\overline{x}) \in S^{\text{qf}}_n(L) \mid \varphi \in p\}\) for all quantifier-free formulas \(\varphi\). Note that \(S^{\text{qf}}_n(L)\) admits an action of the symmetric group \(\text{Sym}(n)\) (the permutation group of \([n]\)), by \(\sigma(p(x_0, \ldots, x_{n-1})) = p(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)})\) for \(\sigma \in \text{Sym}(n)\). We write \(S^{\text{qf}}_n(L)\) for the \(\text{Sym}(n)\)-invariant subspace of non-redundant quantifier-free types, namely those which contain \(x_i \neq x_j\) for all \(i \neq j\).

We let \((\xi_A)_{A \in \mathcal{P}_\text{fin}(\mathbb{N})}\) be a collection of independent random variables, each uniformly distributed on \([0, 1]\). We think of \(\xi_A\) as a source of randomness sitting on the subset \(A\), which we will use to build a random \(L\)-structure with domain \(\mathbb{N}\). If \(\overline{a} \in \mathbb{N}^{[n]}\), the injective function \(i : [n] \rightarrow \mathbb{N}\) enumerating \(\overline{a}\) associates to each \(X \in \mathcal{P}([n])\) a subset \(i[X] \subseteq \|\overline{a}\|\). We denote by \(\widehat{\xi}_{\overline{a}}\) the family of random variables \((\xi_{i[X]} X \in \mathcal{P}([n]))\).

**Definition 2.24.** An **AHK system** is a collection of measurable functions \(f_n : [0, 1]^{\mathcal{P}([n])} \rightarrow S^{\text{qf}}_n(L))_{n \in \mathbb{N}}\) satisfying the coherence conditions:

- For all \(\sigma \in \text{Sym}(n)\), almost surely \[f_n((\xi_{i[X]} X \subseteq [n])) = \sigma(f_n((\xi_{i[X]} X \subseteq [n])).\]

- For all \(m < n\) and \(Z \subseteq [n]\) of size \(m\), almost surely \[f_m((\xi_{i[X]} X \subseteq Z) \subseteq f_n((\xi_{i[Y]} Y \subseteq [n])).\]

That is, \(f_n\) takes as input a collection of values in \([0, 1]\), indexed by \(\mathcal{P}([n])\), and produces a non-redundant quantifier-free \(n\)-type. Using our random variables \(\xi_A\), we have a natural notion of **sampling** from an AHK system to obtain a non-redundant quantifier-free type \(r_{\overline{a}} = f_n(\widehat{\xi}_{\overline{a}})\) for every finite tuple \(\overline{a}\) from \(\mathbb{N}\). Note that the order in which \(\|\overline{a}\|\) is enumerated by the tuple \(\overline{a}\) is significant, since \(f_n\) is, in general, not symmetric in its arguments.

The coherence conditions ensure that the quantifier-free types obtained from the function \(f_n\) cohere (almost surely), allowing us to define the random structure \(\mathfrak{M}\) obtained by sampling from the AHK system \((f_n)\). Namely, for every tuple \(\overline{a} \in \mathbb{N}\),

\[\mathfrak{M} \models R(\overline{a})\quad \text{if and only if} \quad R(\overline{a}) \in f_n(\widehat{\xi}_{\overline{a}}),\]

where \(n\) is the length of \(\overline{a}\).

One may also directly describe the measure on \(\text{Str}_L\) which is the distribution of the random structure \(\mathfrak{M}\): an AHK system \((f_n)_{n \in \mathbb{N}}\) gives rise to a well-defined
finitely-additive probability measure \( \mu^* \) on the Boolean algebra \( B^* \) of clopen sets in \( \text{Str}_L \), defined by

\[
\mu^*(\llbracket \varphi(\overline{a}) \rrbracket) = \lambda^{P([n])}(f_n^{-1}(\llbracket \varphi(\overline{x}) \rrbracket)),
\]

where \( \lambda^{P([n])} \) is the uniform product measure on \( [0, 1]^{P([n])} \). This is the probability that \( \varphi(\overline{x}) \in r_\overline{\pi} \), whenever \( \overline{\pi} \) is a tuple of \( n \) distinct elements. The coherence conditions imply that this is well-defined: the first ensures that the order in which we list the variables in \( \varphi(\overline{\pi}) \) is irrelevant, and the second ensures that the measure is independent of the variable context \( \overline{x} \).

Since the value of \( \mu^*(\llbracket \varphi(\overline{a}) \rrbracket) \) does not depend on the choice of tuple \( \overline{a} \) of distinct elements, \( \mu^* \) is manifestly invariant for the logic action. By Proposition 2.6, \( \mu^* \) induces a unique invariant Borel probability measure \( \mu \) on \( \text{Str}_L \). In this case, we say that \( (f_n)_{n \in \mathbb{N}} \) is an AHK representation of \( \mu \).

**Theorem 2.25** (Aldous–Hoover–Kallenberg, [Kal05, Theorem 7.22]). Every invariant probability measure \( \mu \) on \( \text{Str}_L \) has an AHK representation.

The AHK representation produced by Theorem 2.25 is not unique, but it is unique up to certain appropriately measure-preserving transformations. See [Kal05, Theorem 7.28] for a precise statement.

The key fact to observe about AHK systems is that if \( \overline{a} \) and \( \overline{b} \) are tuples from \( \mathbb{N} \) whose intersection \( \|\overline{a}\| \cap \|\overline{b}\| \) is enumerated by the tuple \( \overline{c} \), then the random quantifier-free types \( r_\overline{a} \) and \( r_\overline{b} \) are conditionally independent over \( \hat{\xi}_\overline{c} \). If \( \overline{a} \) and \( \overline{b} \) are disjoint, then \( \hat{\xi}_\overline{a} = \xi_\emptyset \).

The Aldous–Hoover–Kallenberg theorem also provides a characterization of the ergodic measures among the invariant measures on \( \text{Str}_L \): they are those measures for which the random quantifier-free types \( r_\overline{a} \) and \( r_\overline{b} \) are independent when \( \overline{a} \) and \( \overline{b} \) are disjoint. Formally, for an \( n \)-tuple \( \overline{a} \) from \( \mathbb{N} \), let \( \Sigma_\overline{a} \) be the \( \sigma \)-algebra on \( \text{Str}_L \) generated by the sets \( \llbracket \varphi(\overline{a}) \rrbracket \), where \( \varphi(\overline{a}) \) ranges over the quantifier-free formulas in the \( n \)-tuple of variables \( \overline{a} \). We say that an invariant probability measure \( \mu \) on \( \text{Str}_L \) is dissociated if whenever \( \overline{a} \) and \( \overline{b} \) are disjoint tuples from \( \mathbb{N} \), the \( \sigma \)-algebras \( \Sigma_\overline{a} \) and \( \Sigma_\overline{b} \) are independent (see Remark 2.7 above).

**Theorem 2.26** ([Kal05, Lemma 7.35]). Let \( \mu \) be an invariant probability measure on \( \text{Str}_L \). The following are equivalent:

1. \( \mu \) is ergodic.
2. \( \mu \) is dissociated.
3. \( \mu \) has an AHK representation in which the functions \( f_n \) do not depend on the argument indexed by \( \emptyset \).

The result [Kal05, Lemma 7.35] is stated for finite relational languages, but can be generalized to our setting.
3. Examples

In this section, we give examples of properly ergodic structures. These illustrate some of the key features discussed in the following sections. In the descriptions below, all random choices are made independently, unless otherwise specified. When we say that we pick a random element $A \in 2^\mathbb{N}$, we always refer to the natural (Lebesgue) measure on $2^\mathbb{N}$, the infinite product of the Bernoulli($\frac{1}{2}$) measure on $2 = \{0, 1\}$. We identify such an $A \in 2^\mathbb{N}$ with both a subset of $\mathbb{N}$ and an infinite binary sequence.

**Example 3.1 (Kaleidoscopes).** Our first example, called the **kaleidoscope random graph**, comes from a class of examples introduced in [AFNP16]. Let $L$ be the language $\{R_n | n \in \mathbb{N}\}$, where each $R_n$ is a binary relation symbol. The interpretation of each $R_n$ will be anti-reflexive and symmetric.

We build an $L$-structure with domain $\mathbb{N}$, by picking a random $A_{\{i,j\}} \in 2^\mathbb{N}$ for each pair $\{i,j\}$ from $\mathbb{N}$ and setting $iR_nj$ if and only if $n \in A_{\{i,j\}}$. This construction describes an ergodic structure $\mu$: $\mu$ is invariant, since the random quantifier-free type of a tuple of distinct elements does not depend on the choice of tuple, and $\mu$ is ergodic, since the random quantifier-free types of disjoint tuples are independent (Theorem 2.26). Note that there are continuum-many possible quantifier-free 2-types, each of which is realized with probability 0. Hence any particular countable $L$-structure, which must realize some countable collection of these types, appears with probability 0 up to isomorphism, so $\mu$ is properly ergodic.

In fact, for every $A \in 2^\mathbb{N}$, the theory $\text{Th}(\mu)$ contains the sentence

$$\neg \exists x \exists y \left( \bigwedge_{n \in A} xR_ny \land \bigwedge_{n \notin A} \neg xR_ny \right).$$

Since all possible quantifier-free 2-types are ruled out by $\text{Th}(\mu)$, this theory has no models of any cardinality. Note, however, that any countable fragment $F$ of $L_{\omega_1,\omega}$ only contains countably many of the sentences above, so $\text{Th}_F(\mu)$ only rules out countably many of the quantifier-free 2-types.

Restricting to the first-order fragment, the theory $\text{Th}_{\text{FO}}(\mu)$ has several nice properties. It is the model companion of the universal theory asserting that each $R_n$ is anti-reflexive and symmetric. It can be axiomatized by extension axioms, analogous to those in the theory of the Rado graph, asserting, for each finite sublanguage $L^* \subseteq L$, that any finite set of points $A$ can be extended by a new element $b$ with an arbitrary non-redundant quantifier-free type over $A$ in the language $L^*$. The reduct of $\text{Th}_{\text{FO}}(\mu)$ to any finite sublanguage is countably categorical, but $\text{Th}_{\text{FO}}(\mu)$ has continuum-many countable models (clearly, since there are continuum-many quantifier-free 2-types consistent with $\text{Th}_{\text{FO}}(\mu)$).
This example can be modified to produce the **kaleidoscope random n-hypergraph** for any \( n \). We call the case \( n = 1 \) the **kaleidoscope random predicate**.

**Example 3.2 (Random geometric graphs).** Bonato and Janssen [BJ11] introduced a new geometric random graph model: given a countable dense set \( V \) of points in some metric space \( (S,d) \), and a real number \( p \in (0,1) \), construct a graph on \( V \) by connecting \( x \) and \( y \) by an edge with probability \( p \), independently for each pair \( \{x,y\} \) from \( V \) such that \( d(x,y) \leq 1 \). If, in addition, we obtain the set \( V \) by i.i.d. sampling from some distribution on \( S \) with a strictly positive density function, then this random construction describes an ergodic structure, called the **random geometric graph on** \( S \) **with edge probability** \( p \) (we suppress the distribution on \( S \), since the particular choice of distribution turns out to be irrelevant).

Bonato and Janssen showed that if we take \( S \) to be \( \ell^n_\infty \) for some \( n \), then the random geometric graph on \( S \) with edge probability \( p \) is almost surely isomorphic to a single countable graph, denoted \( \text{GR}_n \). Later, Balister, Bollobás, Gunderson, Leader, and Walters [BBG+15] showed that the spaces \( \ell^n_\infty \) are the unique normed spaces with this property: if \( S \) is a normed space that is not isometric to \( \ell^n_\infty \) for any \( n \), then the random geometric graph on \( S \) with edge probability \( p \) is properly ergodic.

The next example illustrates the distinction between group-theoretic and syntactic definable closure.

**Example 3.3 (Trivial dcl).** Let \( T \) be the first-order theory of the kaleidoscope random predicate (see Example 3.1) in the language \( \{P_n \mid n \in \mathbb{N}\} \). The theory \( T \) says that for every \( m \in \mathbb{N} \) and every subset \( A \subseteq [m] \), there is an element \( x \) such that for all \( n \in [m] \), \( P_n(x) \) holds if and only if \( n \in A \).

Now let \( T' \) be \( T \) together with the infinitary sentence
\[
\forall x \forall y (x \neq y) \rightarrow \bigvee_{n \in \mathbb{N}} \neg(P_n(x) \leftrightarrow P_n(y)). 
\]

The kaleidoscope random predicate almost surely satisfies \( T' \). Each of the continuum-many quantifier-free 1-types is realized with probability 0, and since the quantifier-free 1-types of distinct elements of \( \mathbb{N} \) are independent, almost surely no 1-type is realized more than once.

In a model \( M \) of \( T' \), no two elements have the same quantifier-free 1-type. Hence \( \text{Aut}(M) \) is the trivial group, and \( M \) has non-trivial group-theoretic dcl. But the countable fragment of \( \mathcal{L}_{\omega_1,\omega} \) generated by \( T' \) does not contain the conjunctions of the form \( \bigwedge_{n \in A} P_n(x) \land \bigwedge_{n \notin A} \neg P_n(x) \) for \( A \subseteq \mathbb{N} \) needed to pin down elements uniquely. In fact, any completion of \( T' \) to an \( F \)-theory has trivial dcl, by Theorem 2.17.
We will see that the presence of a formula $\chi(\bar{x})$ of positive measure such that every type containing $\chi$ has probability 0 of being realized, is a characteristic feature of properly ergodic structures. In the kaleidoscope random graph (Example 3.1), $x \neq y$ is such a formula, since every non-redundant quantifier-free 2-type is realized with probability 0. In contrast to the kaleidoscope random graph, Example 3.4 shows that these 0-probability types may have infinitely many realizations if they are realized at all.

On the other hand, in Example 3.5, we see that requiring each of a family of continuum-many 1-types to be realized infinitely many times (if at all) can rule out proper ergodicity. This shows that the consistency of continuum-many types in a theory with trivial dcl is not sufficient for the existence of a properly ergodic model of the theory.

These phenomena motivate the definition of rootedness in Section 5.

**Example 3.4** (The max graph). As in Example 3.1, let $L = \{R_n \mid n \in \mathbb{N}\}$, where each $R_n$ is a binary relation symbol. We build a random $L$-structure with domain $\mathbb{N}$ such that the interpretation of each $R_n$ is anti-reflexive and symmetric. For each $i \in \mathbb{N}$, choose a random element $A_i \in 2^\mathbb{N}$. Now for each pair $\{i, j\}$, let $A_{ij} = \max(A_i, A_j)$, where we give $2^\mathbb{N}$ its lexicographic order. We set $iR_nj$ if and only if $n \in A_{ij}$.

We have continuum-many quantifier-free 2-types $\{p_A \mid A \in 2^\mathbb{N}\}$, where $xR_ny \in p_A$ if and only if $n \in A$, and each is realized with probability 0, since if $(i, j)$ realizes $p_A$, we must have $A_i = A$ or $A_j = A$.

As long as $A_i$ is not the constant 0 sequence (which appears with probability 0), then for any $j \neq i$, there is a positive probability, conditioned on the choice of $A_i$, that $A_j \leq A_i$, and hence $qftp(i, j) = p_{A_i}$. Since the $A_j$ are chosen independently, almost surely the event $A_j \leq A_i$ occurs for infinitely many $j$. So, almost surely, any non-redundant quantifier-free 2-type that is realized is realized infinitely many times. However, since the probability that $A_i = A_j$ when $i \neq j$ is 0, almost surely all realizations of $p_{A_i}$ have a common intersection, namely the vertex $i$.

**Example 3.5** (A nonexample). Let $L = \{E\} \cup \{P_n \mid n \in \mathbb{N}\}$, and let $T$ be the model companion of the universal theory asserting that $E$ is an equivalence relation and the $P_n$ are unary predicates respecting $E$ (if $xEy$, then $P_n(x)$ if and only if $P_n(y)$). This is similar to the first-order theory of the kaleidoscope random predicate, but with each element replaced by an infinite $E$-class.

There is no properly ergodic structure that satisfies $T$ almost surely. Indeed, suppose $\mu \models T$. Then for every quantifier-free 1-type $p$, there is some probability $\mu(p)$ that $p$ is the quantifier-free type of the element $i \in \mathbb{N}$, and, by invariance, $\mu(p)$ does not depend on the choice of $i$. We denote by $S_{qf}^1(\mu)$ the set of quantifier-free 1-types with positive measure. If $\sum_{p \in S_{qf}^1(\mu)} \mu(p) = 1$, then almost surely only the types in $S_{qf}^1(\mu)$ are realized, since $\mu \models \forall x \bigvee_{p \in S_{qf}^1(\mu)} \varphi(x)$. Further, $\mu$ determines, for each $p \in S_{qf}^1(\mu)$, the number of $E$-classes on which $p$ is realized.
(among \{1, 2, \ldots, R_0\}), since each of the countably many choices is expressible by a sentence of \(L_{\omega_1, \omega}\). The data of which quantifier-free 1-types are realized, and how many \(L\)-classes realize each, determines a unique \(L\)-structure up to isomorphism, so \(\mu\) is not properly ergodic.

On the other hand, if \(\sum_{p \in S^1_{qf}(\mu)} \mu(p) < 1\), then almost surely some types that are not in \(S^1_{qf}(\mu)\) are realized. Any such type \(p\) is realized with probability 0, and, by ergodicity, the quantifier-free 1-types of distinct elements of \(N\) are independent. So, almost surely, each of the 0-probability types is realized at most once. This contradicts the fact that any realized type must be realized on an entire infinite \(L\)-class.

**Example 3.6 (The necessity of infinitary logic).** Let \(L = \{P\} \cup \{R^i_j \mid i, j \in \mathbb{N}\}\), where \(P\) is a unary predicate and the \(R^i_j\) are binary relations, and let \(T\) be the model companion of the following universal theory:

1. \(\forall x \forall y R^i_j(x, y) \rightarrow (P(x) \land \neg P(y))\) for all \(i\) and \(j\).
2. \(\forall x \forall y \neg R^0_0(x, y) \land R^i_0(x, y)\) for all \(i \neq i'\).
3. \(\forall x \forall y R^i_{j+1}(x, y) \rightarrow R^i_j(x, y)\) for all \(i\) and \(j\).

Thus, a model of \(T\) is a bipartite graph in which each edge from \(x\) to \(y\) is labeled by some \(i \in \mathbb{N}\) (in the superscript) and the set of all \(j < k\) for some \(k \in \mathbb{N}_+ \cup \{\infty\}\) (in the subscript), where \(\mathbb{N}_+\) denotes the positive natural numbers.

Now \(T\) is a complete theory with quantifier elimination and with only countably many types over \(\emptyset\). Hence, by countable additivity, if \(\mu\) is an ergodic structure that satisfies \(T\) almost surely, then there is no positive-measure first-order formula \(\chi(\bar{x})\) such that every type containing \(\chi\) has measure 0. Nevertheless, we will describe a properly ergodic structure that almost surely satisfies \(T\).

First, for each \(x \in \mathbb{N}\), set \(P(x)\) with probability \(1/2\), and pick \(A_x \in 2^\mathbb{N}\) at random. Now for each pair \(x \neq y\), if \(P(x)\) and \(\neg P(y)\), then we choose which of the \(R^i_j\) to assign to \((x, y)\). Choose \(i \in \mathbb{N}\) at random, setting \(i = n\) with probability \(2^{-(n+1)}\). If \(i \in A_x\), choose \(k \in \mathbb{N}_+ \cup \{\infty\}\) at random, setting \(k = \infty\) with probability \(1/2\) and \(k = n\) with probability \(2^{-(n+1)}\) for \(n \in \mathbb{N}_+\). If \(i \notin A_x\), choose \(k \in \mathbb{N}_+\) at random, setting \(k = n\) with probability \(2^{-n}\). Finally, set \(R^i_j(x, y)\) for all \(j < k\).

In the resulting random structure, we can almost surely recover \(A_x\) from every \(x \in P\), since if \(i \in A_x\), then almost surely there is some \(y\) such that \(R^i_j(x, y)\) for all \(j \in \mathbb{N}\) (that is, the choice \(k = \infty\) was made for the pair \((x, y)\)), whereas this outcome is impossible if \(i \notin A_x\). Thus the structure encodes a countable set of elements of \(2^\mathbb{N}\), each of which occurs with probability 0.

The information encoding \(A_x\) is part of the 1-type of \(x\) in any countable fragment of \(L_{\omega_1, \omega}\) containing the infinitary formulas \(\{\exists y \land_{j \in \mathbb{N}} R^i_j(x, y) \mid i \in \mathbb{N}\}\), but it is not expressible in first-order logic.
With the exception of Example 3.2, the preceding examples have all used infinite languages, as this is the easiest setting in which to split the measure over continuum-many types. We conclude with an elementary example in the language with a single binary relation.

**Example 3.7 (An example in a finite language)**. Let \( L = \{ R \} \), where \( R \) is a binary relation. In our probabilistic construction, we will enforce the following almost surely:

- Let \( O = \{ x \mid R(x,x) \} \), and \( P = \{ x \mid \neg R(x,x) \} \). Then \( O \) and \( P \) are both infinite sets.
- If \( R(x,y) \), then either \( x \) and \( y \) are both in \( O \), or \( x \) is in \( P \) and \( y \) is in \( O \).
- \( R \) is a preorder on \( O \). Denote by \( xEy \) the induced equivalence relation \( R(x,y) \land R(y,x) \). Then \( E \) has infinitely many infinite classes, and \( R \) linearly orders the \( E \)-classes with order type \( \omega \).
- Given \( x \in P \) and \( y, z \in O \), if \( R(x,y) \) and \( yEz \), then \( R(x,z) \). So \( R \) relates each element of \( P \) to some subset of the \( E \)-classes.

Thus we can interpret the kaleidoscope random predicate on \( P \), where the \( n^{th} \) predicate \( P_n \) holds of \( x \) if and only if \( x \) is \( R \)-related to the \( n^{th} \) class in the linear order on \( O \).

Now it is straightforward to describe the probabilistic construction: for each \( i \in \mathbb{N} \), let \( R(i,i) \) hold with probability \( 1/2 \). This determines whether \( i \) is in \( O \) or \( P \). If \( i \in O \), we choose which \( E \)-class to put \( i \) in, under the order induced by \( R \), selecting the \( n^{th} \) class with probability \( 2^{-(n+1)} \). These choices determine all the \( R \)-relations between elements of \( O \). On the other hand, if \( x \in P \), we pick \( A_i \in 2^{\mathbb{N}} \) at random and relate \( i \) to each the \( n^{th} \) class in \( O \) if and only if \( n \in A_i \).

This describes an ergodic structure \( \mu \), since the quantifier-free types of disjoint tuples are independent. We obtain the properties described in the bullet points above almost surely, and since \( \omega \) is rigid, any isomorphism between structures satisfying these properties must preserve the order on the \( E \)-classes. For any subset of the \( E \)-classes, the probability is 0 that there is an element of \( P \) which is related to exactly those \( E \)-classes, and so \( \mu \) is properly ergodic.

## 4. Morley–Scott analysis of ergodic structures

Throughout this section, let \( \mu \) be an ergodic structure. Recall from Section 2 that if \( F \) is a countable fragment of \( \mathcal{L}_{\omega_1,\omega} \) and \( p \) is an \( F \)-type, we use the abbreviation \( \theta_p(\bar{x}) \) to mean \( \bigwedge_{\varphi \in p} \varphi(\bar{x}) \) and \( \mu(p) \) to mean \( \mu(\theta_p(\bar{x})) \).

**Definition 4.1.** We denote by \( S^n_F(\mu) \) the set \( \{ p \mid \mu(p) > 0 \} \) of **positive-measure \( F \)-types** in the variables \( x_0, \ldots, x_{n-1} \). We include the case \( n = 0 \): \( S^0_F(\mu) \) has one element, namely \( \text{Th}_F(\mu) \).

**Lemma 4.2.** For all \( n \in \mathbb{N} \), \( |S^n_F(\mu)| \leq \aleph_0 \).
Proof. Fix a tuple $\bar{a}$ of distinct elements from $\omega$. The sets $\{[\theta_p(\bar{a})] \mid p \in S^n_F(\mu)\}$ are disjoint sets of positive measure in $\text{Str}_L$. By additivity of $\mu$, for all $m \in \mathbb{N}$, $P_m = \{p \in S^n_F(\mu) \mid \mu(p) \geq 1/m\}$ is finite (of size at most $m$), so $S^n_F(\mu) = \bigcup_{m \in \omega} P_m$ is countable.

We build a sequence $\{F_\alpha\}_{\alpha \in \omega_1}$ of countable fragments of $L_{\omega_1, \omega}$ of length $\omega_1$, depending on the ergodic structure $\mu$:

$$F_0 = \text{FO},$$
the first-order fragment.

$$F_{\alpha+1} = \text{the fragment generated by } F_\alpha \cup \left\{ \theta_p(\bar{x}) \mid p \in \bigcup_{n \in \mathbb{N}} S^n_{F_\alpha}(\mu) \right\}.$$

$$F_\gamma = \bigcup_{\alpha < \gamma} F_\alpha, \text{ if } \gamma \text{ is a limit ordinal.}$$

**Definition 4.3.** We say that $p \in S^n_{F_\alpha}(\mu)$ splits at $\beta > \alpha$ if $\mu(q) < \mu(p)$ for all types $q \in S^n_{F_\beta}(\mu)$ such that $p \subseteq q$. We say that $p$ splits later if there exists $\beta$ such that $p$ splits at $\beta$. We say that $\mu$ has stabilized at $\gamma$ if for all $n \in \mathbb{N}$, no type in $S^n_{F_\gamma}(\mu)$ splits later.

**Lemma 4.4.** Let $\alpha < \beta < \gamma$.

1. If a type $p \in S^n_{F_\alpha}(\mu)$ splits at $\beta$, then $p$ also splits at $\gamma$.
2. Suppose $p \in S^n_{F_\alpha}(\mu)$ splits at $\gamma$. Then $p' = p \cap F_\alpha$ is in $S^n_{F_\beta}(\mu)$ and also splits at $\gamma$.
3. If no type in $S^n_{F_\alpha}(\mu)$ splits later, then no type in $S^n_{F_\beta}(\mu)$ splits later.

**Proof.**
1. Pick $q \in S^n_{F_\alpha}(\mu)$ with $p \subseteq q$, and let $q' = q \cap F_\beta$. Then $\mu(q) \leq \mu(q') < \mu(p)$, since $p$ splits at $\beta$.
2. First, $0 < \mu(p) \leq \mu(p')$, so $p' \in S^n_{F_\alpha}(\mu)$. Pick $q \in S^n_{F_\alpha}(\mu)$ such that $p' \subseteq q$. If $p \subseteq q$, then $\mu(q) < \mu(p) \leq \mu(p')$, since $p$ splits at $\gamma$. And if $p \not\subseteq q$, then $\mu(q) \leq \mu(p') - \mu(p) < \mu(p')$, since $\mu(p) > 0$. In either case, $\mu(q) < \mu(p')$, so $p'$ splits at $\gamma$.
3. If some type in $S^n_{F_\beta}(\mu)$ splits later, then by (2), $p' = p \cap F_\alpha$ also splits later, and $p' \in S^n_{F_\gamma}(\mu)$.

**Lemma 4.5.** There is some countable ordinal $\gamma$ such that $\mu$ has stabilized at $\gamma$.

**Proof.** Fix $n \in \mathbb{N}$. For each $\alpha \in \omega_1$, let

$$\text{Sp}(\alpha) = \{p \in S^n_{F_\alpha}(\mu) \mid p \text{ splits later}\},$$

$$r_\alpha = \sup\{\mu(p) \mid p \in \text{Sp}(\alpha)\}.$$

Note that $\text{Sp}(\alpha)$ is countable, since $S^n_{F_\alpha}$ is. If $\text{Sp}(\alpha)$ is non-empty, then $r_\alpha > 0$, and in fact the supremum is achieved by finitely many types, since $\sum_{p \in \text{Sp}(\alpha)} \mu(p) \leq 1$. 

□
By Lemma 4.4 (2), the measure of any type in \( \text{Sp}(\beta) \) is bounded above by the measure of a type in \( \text{Sp}(\alpha) \), namely its restriction to \( F_\alpha \). So we have \( r_\beta \leq r_\alpha \) whenever \( \alpha < \beta \).

Now assume for contradiction that \( \text{Sp}(\alpha) \) is non-empty for all \( \alpha \). We build a strictly increasing sequence \( \langle \alpha_\delta \rangle_{\delta \in \omega_1} \) in \( \omega_1 \), such that \( \langle r_{\alpha_\delta} \rangle_{\delta \in \omega_1} \) is a strictly decreasing sequence in \([0, 1]\). Begin with \( \alpha_0 = 0 \).

At each successor stage, we are given \( \alpha = \alpha_\delta \), and we seek \( \beta = \alpha_{\delta+1} \) with \( r_\beta < r_\alpha \). Since \( \text{Sp}(\alpha) \) is non-empty, there are finitely many types \( p_1, \ldots, p_n \) of maximal measure \( r_\alpha > 0 \). For each \( i \), pick \( \beta_i > \alpha \) such that \( p_i \) splits at \( \beta_i \), and let \( \beta = \max(\beta_1, \ldots, \beta_n) \). By Lemma 4.4 (1), each \( p_i \) splits at \( \beta \). Let \( q \) be a type in \( \text{Sp}(\beta) \) with \( \mu(q) = r_\beta \), and let \( q' = q \cap F_\alpha \). By Lemma 4.4 (2), \( q' \in \text{Sp}(\alpha) \).

If \( q' \) is one of the \( p_i \), then \( \mu(q) < \mu(p_i) = r_\alpha \), since \( p_i \) splits at \( \beta \). If not, then \( \mu(q) \leq \mu(q') < r_\alpha \). In either case, \( r_\beta = \mu(q) < r_\alpha \).

If \( \lambda \) is a countable limit ordinal, let \( \alpha_\lambda = \sup_{\delta < \lambda} \alpha_\delta \). This is an element of \( \omega_1 \), since \( \omega_1 \) is regular. And for all \( \delta < \lambda \), since \( \alpha_{\delta+1} < \alpha_\lambda \), we have \( r_{\alpha_\lambda} \leq r_{\alpha_{\delta+1}} < r_{\alpha_\delta} \).

Of course, there is no strictly decreasing sequence of real numbers of length \( \omega_1 \), since \( \mathbb{R} \) contains a countable dense set. Hence there is some \( \gamma_n \in \omega_1 \) such that \( \text{Sp}(\gamma_n) \) is empty, i.e., no type in \( \text{Sp}_{F_{\gamma_n}} \) splits later. Let \( \gamma = \sup_{n \in \mathbb{N}} \gamma_n \in \omega_1 \). Then by Lemma 4.4 (3), \( \mu \) has stabilized at \( \gamma \).

We can think of the minimal ordinal \( \gamma \) such that \( \mu \) has stabilized at \( \gamma \) as an analogue of the Scott rank for the ergodic structure \( \mu \). Since no \( F_\gamma \)-type splits later, every positive-measure \( F_{\gamma+1} \)-type \( q \) is isolated by the \( F_{\gamma+1} \)-formula \( \theta_p \) for its restriction \( p = q \cap F_\gamma \), relative to \( \text{Th}_{F_{\gamma+1}}(\mu) \). Lemma 4.6 says that if every tuple satisfies one of these positive-measure types almost surely, then \( \mu \) almost surely satisfies a Scott sentence.

**Lemma 4.6.** Suppose that \( \mu \) has stabilized at \( \gamma \), and that for all \( n \in \mathbb{N} \),

\[
\sum_{p \in S_{F_{\gamma}}^{\mu}(p)} \mu(p) = 1.
\]

Then \( \mu \) concentrates on a countable structure.

**Proof.** For each type \( r(\varphi) \in S_{F_{\gamma}}^{\mu}(\mu) \) (we include the case \( n = 0 \)), let \( E_r \) be the set of types \( q(\varphi, y) \in S_{F_{\gamma}}^{n+1}(\mu) \) with \( r \subseteq q \). Fix a type \( p(\varphi) \in S_{F_{\gamma}}^{\mu}(\mu) \), let \( \varphi_p \) be the sentence

\[
\forall \varphi \left( \theta_p(\varphi) \to \forall (y \notin \varphi) \bigvee_{q \in E_p} \theta_q(\varphi, y) \right),
\]

and let \( \psi_p \) be the sentence

\[
\forall \varphi \left( \theta_p(\varphi) \to \bigwedge_{q \in E_p} \exists (y \notin \varphi) \theta_q(\varphi, y) \right).
\]
Here \(\forall (y \not\in \overline{x}) \rho(\overline{x}, y)\) and \(\exists (y \not\in \overline{x}) \rho(\overline{x}, y)\) are shorthand for \(\forall y((\bigwedge_{i=0}^{n-1} y \neq x_i) \rightarrow \rho(\overline{x}, y))\) and \(\exists y((\bigwedge_{i=0}^{n-1} y \neq x_i) \land \rho(\overline{x}, y))\), respectively. We would like to show that \(\mu\) satisfies \(\varphi_p\) and \(\psi_p\) almost surely.

By assumption, and since every \(q \in S_{F_{\overline{p}}}^{n+1}(\mu)\) is in \(E_r\) for a unique \(r \in S_{F_{\overline{p}}}^n(\mu)\),

\[
1 = \sum_{q \in S_{F_{\overline{p}}}^{n+1}(\mu)} \mu(q) = \sum_{r \in S_{F_{\overline{p}}}^n(\mu)} \sum_{q \in E_r} \mu(q).
\]

Then for all \(r \in S_{F_{\overline{p}}}^n(\mu)\), we must have

\[
\mu(r) = \sum_{q \in E_r} \mu(q).
\]

In particular, this is true for \(r = p\), so for any tuple \(\overline{a}\) and any \(b\) not in \(\overline{a}\), \([\bigvee_{q \in E_p} \theta_q(\overline{a}, b)]\) has full measure in \([\theta_p(\overline{a})]\) (this is true even when \(\overline{a}\) contains repeated elements, since in that case \([\theta_p(\overline{a})]\) has measure 0). A countable intersection (over \(b \in \mathbb{N} \setminus \|\overline{a}\|\)) of subsets of \([\theta_p(\overline{a})]\) with full measure still has full measure, so

\[
\mu([\theta_p(\overline{a}) \rightarrow \forall (y \not\in \overline{a}) \bigvee_{q \in E_p} \theta_q(\overline{a}, y)]) = 1.
\]

Taking another countable intersection over all tuples \(\overline{a}\), we have \(\mu \models \varphi_p\).

We turn now to \(\psi_p\). Since \(\mu\) stabilizes at \(\gamma\), there is a (necessarily unique) extension of \(p\) to a type \(p^* \in S_{F_{\overline{p}}}^n(\mu)\) with \(\mu(p^*) = \mu(p)\). Let \(q(\overline{x}, y)\) be any type in \(E_p\), and let \(v_q(\overline{x}) \in F_{\overline{p}} \gamma+1\) be the formula \(\exists y \not\in \overline{x} \theta_q(\overline{x}, y)\). Note that \(\theta_q(\overline{x}, y)\) implies \(v_q(\overline{x})\) and \(v_q(\overline{x})\) implies \(\theta_p(\overline{x})\). So \(\mu(v_q(\overline{x})) \geq \mu(q) > 0\), and we must have \(v_q(\overline{x}) \in p^*\), otherwise \(\mu(p^*) \leq \mu(p) - \mu(v_q(\overline{x}))\). Finally, we conclude that for any tuple \(\overline{a}\), \([v_q(\overline{a})]\) has full measure in \([\theta_p(\overline{a})]\), since \(\mu(p) = \mu(p^*) \leq \mu(v_q(\overline{x})) \leq \mu(p)\).

As before, a countable intersection of subsets with full measure has full measure, so

\[
\mu([\theta_p(\overline{a}) \rightarrow \bigwedge_{q \in E_p} \exists (y \not\in \overline{a}) \theta_q(\overline{a}, y)]) = 1.
\]

Taking another countable intersection over all tuples \(\overline{a}\), we have \(\mu \models \psi_p\).

Let \(T = \text{Th}_{F_{\overline{p}}}(\mu) \cup \{\varphi_p, \psi_p \mid p \in \bigcup_{n \in \mathbb{N}} S_{F_{\overline{p}}}^n(\mu)\}\), and note that \(T\) is countable. Since \(\mu\) almost surely satisfies \(T\), it suffices to show that any two countable models of \(T\) are isomorphic. This is a straightforward back-and-forth argument, using \(\varphi_p\) and \(\psi_p\) to extend a partial \(F_{\overline{p}}\)-elementary isomorphism defined on a realization of \(p\) by one step: \(\varphi_p\) tells us that each one-point extension in one model realizes one of the types in \(E_p\), and \(\psi_p\) tells us that every type in \(E_p\) is realized in a one-point extension in the other model. To start, the empty tuples in any two models of \(T\) satisfy the same \(F_{\overline{p}}\)-type, namely \(\text{Th}_{F_{\overline{p}}}(\mu)\). \(\square\)
Theorem 4.7. Let $\mu$ be an ergodic structure. Then $\mu$ is properly ergodic if and only if for every countable fragment $F$ of $\mathcal{L}_{\omega_1, \omega}$, there is a countable fragment $F' \supseteq F$ and a formula $\chi(\bar{x})$ in $F'$ such that $\mu(\chi(\bar{x})) > 0$, but $\mu(p) = 0$ for every $F'$-type $p(\bar{x})$ containing $\chi(\bar{x})$.

Proof. Suppose $\mu$ is properly ergodic. By Lemma 4.5, $\mu$ stabilizes at some $\gamma$, and by Lemma 4.6, there is some $n$ such that $\sum_{p \in S^n_{\gamma}(\mu)} \mu(p) = 1$. Let $\chi(\bar{x})$ be the formula $\bigwedge_{p \in S^n_{\gamma}(\mu)} \neg \theta_p(\bar{x})$. Then $\mu(\chi(\bar{x})) > 0$.

Let $F'$ be the countable fragment generated by $F \cup F' \cup \{\chi(\bar{x})\}$, and suppose that $p(\bar{x})$ is an $F'$-type containing $\chi(\bar{x})$. Let $q = p \cap F'$. Then $q$ is an $F'_{\gamma}$ type that is consistent with $\chi(\bar{x})$, so $q \notin S^n_{\gamma}(\mu)$, and $\mu(p) \leq \mu(q) = 0$.

Conversely, suppose we have such a fragment $F'$ and such a formula $\chi(\bar{x})$. Since $\mu(\chi(\bar{x})) > 0$, by ergodicity, $\mu \models \exists \bar{x} \chi(\bar{x})$. Let $M$ be a countable structure. If $M$ contains no tuple satisfying $\chi$, then $\mu$ assigns measure 0 to the isomorphism class of $M$. On the other hand, if $M$ contains a tuple $\bar{a}$ satisfying $\chi(\bar{x})$, then since $\mu$ assigns measure 0 to the set of structures realizing $tp_{F'}(\bar{a})$, it also assigns measure 0 to the isomorphism class of $M$. So $\mu$ is properly ergodic. \[\square\]

By countable additivity, if a sentence $\varphi$ of $\mathcal{L}_{\omega_1, \omega}$ has only countably many countable models up to isomorphism, then any ergodic structure $\mu$ that almost surely satisfies $\varphi$ is almost surely isomorphic to one of its models. That is, no ergodic model of $\varphi$ is properly ergodic. We show now that the same is true if $\varphi$ is a counterexample to Vaught’s conjecture, i.e., a sentence with uncountably many, but fewer than continuum-many, countable models.

Corollary 4.8 (“Vaught’s Conjecture for ergodic structures”). Let $\varphi$ be a sentence of $\mathcal{L}_{\omega_1, \omega}$. If there is a properly ergodic structure $\mu$ such that $\mu \models \varphi$, then $\varphi$ has continuum-many countable models up to isomorphism.

Proof. This is a consequence of Theorem 4.7 and an observation due to Morley [Mor70]: for any countable fragment $F$ of $\mathcal{L}_{\omega_1, \omega}$ containing $\varphi$ and any $n \in \mathbb{N}$, the set $S^n_F(\varphi)$ of $F$-types consistent with $\varphi$ is an analytic subset of $2^F$. Since analytic sets have the Perfect Set Property, if $|S^n_F(\varphi)| > \aleph_0$, then $|S^n_F(\varphi)| = 2^{\aleph_0}$. And since a countable structure realizes only countably many $n$-types, if $|S^n_F(\varphi)| = 2^{\aleph_0}$, then $\varphi$ must have continuum-many countable models up to isomorphism.

Now let $\mu$ be the given proper ergodic structure, let $F$ be a countable fragment containing $\varphi$, let $F'$ and $\chi(\bar{x})$ be as in Theorem 4.7, let $n$ be the length of the tuple $\bar{x}$, and suppose for a contradiction that $|S^n_{F'}(\varphi)| \leq \aleph_0$. Let $U_\chi = \{p \in S^n_{F'}(\varphi) \mid \chi(\bar{x}) \in p\}$. Then $U_\chi$ is countable, and, by our choice of $\chi(\bar{x})$, we have $\mu(p) = 0$ for all $p \in U_\chi$. Since $\mu(\llbracket \varphi \rrbracket) = 1$, for any tuple $\bar{a}$ of distinct elements of $\mathbb{N}$, we have

$$0 < \mu(\llbracket \chi(\bar{a}) \rrbracket) = \mu(\llbracket (\varphi \land \chi)(\bar{a}) \rrbracket) = \mu \left( \bigcup_{p \in U_\chi} \llbracket \theta_p(\bar{a}) \rrbracket \right) = \sum_{p \in U_\chi} \mu(p),$$
which is a contradiction, by countable additivity of \( \mu \).

Kechris has observed (in private communication) that Corollary 4.8 also follows from a result in descriptive set theory [Kec95, Exercise 17.14]: an analogue for measure of a result of Kuratowski about category [Kur76]. However, our proof above provides additional model-theoretic information about properly ergodic structures.

Recall that \( \text{Th}(\mu) \) is the complete \( L_{\omega_1,\omega} \)-theory of \( \mu \). As noted in Remark 2.11, \( \mu \) is properly ergodic if and only if \( \text{Th}(\mu) \) has no countable models. In fact, if \( \mu \) is properly ergodic, then \( \text{Th}(\mu) \) has no models at all. This is stronger, since the Löwenheim–Skolem theorem fails for complete theories of \( L_{\omega_1,\omega} \).

Corollary 4.9. If \( \mu \) is properly ergodic, then \( \text{Th}(\mu) \) has no models (of any cardinality). However, for any countable fragment \( F \) of \( L_{\omega_1,\omega} \), \( \text{Th}_F(\mu) \) has continuum-many countable models up to isomorphism.

Proof. Starting with any countable fragment \( F \) (e.g., \( F = FO \)), let \( F' \) and \( \chi(\bar{x}) \) be as in Theorem 4.7. Then \( \mu(\chi(\bar{x})) > 0 \), so by ergodicity, \( \exists \bar{x} \chi(\bar{x}) \in \text{Th}(\mu) \). Now if \( \text{Th}(\mu) \) has a model \( M \), then there is some tuple \( \bar{a} \) from \( M \) satisfying \( \chi(\bar{x}) \). Let \( p \) be the \( F' \)-type of \( \bar{a} \). Since \( p \) contains \( \chi(\bar{x}) \), we have \( \mu(p) = 0 \), and so \( \neg \exists \bar{x} \theta_p(\bar{x}) \in \text{Th}(\mu) \), contradiction.

The last assertion follows from Corollary 4.8, taking \( \varphi = \bigwedge_{\psi \in \text{Th}_F(\mu)} \psi \).

5. Rooted models

The Morley–Scott analysis in Section 4 showed that proper ergodicity of \( \mu \) can always be explained by a positive-measure formula \( \chi(\bar{x}) \) such that any type containing \( \chi(\bar{x}) \) has measure 0. In a countable structure sampled from \( \mu \), each of these types of measure 0 will be realized “rarely”. Sometimes “rarely” means “at most once”, as in Examples 3.1 and 3.3. But in Example 3.4 we saw that a type \( p \) of measure 0 can be realized by infinitely many tuples, all of which share a common element \( i \in \mathbb{N} \). This is explained by the fact that \( p \) has positive measure after conditioning on a random choice “living at” some finite set containing \( i \). In this section, we will use the AHK theorem from Section 2.4 to show that this behavior, which we call rootedness, is typical.

Throughout this section, let \( F \) be a countable fragment of \( L_{\omega_1,\omega_1} \), and let \( T \) be an \( F \)-theory. We write \( S^{[n]}_F(T) \) for the subspace of \( S^n_F(T) \) consisting of non-redundant \( F \)-types, those which contain \( x_i \neq x_j \) for all \( i \neq j \).

Definition 5.1. Let \( p \in S^{[n]}_F(T) \) be a type realized in \( M \models T \). An element \( a \in M \) is called a root of \( p \) in \( M \) if \( a \) is an element of every tuple realizing \( p \) in \( M \). We use the same terminology for quantifier-free types in \( S^{[n]}_{qf}(T) \).

Remark 5.2. If a type \( p \) has a unique realization in \( M \), then \( p \) has a root in \( M \) (take any element of the unique tuple realizing \( p \)). When \( n = 1 \), the converse is
true: a realized type $p(x) \in S^{[1]}_{F}(T)$ (or $S^{[1]}_{q}(T)$) has a root in $M$ if and only if it has a unique realization in $M$.

**Definition 5.3.** Let $\chi(\bar{x})$ is a formula in $F$ such that $\chi(\bar{x}) \land (\bigwedge_{i \neq j} x_i \neq x_j)$ is consistent with $T$. Then a model $M \models T$ is $\chi$-rooted if every type $p(\bar{x}) \in S^{[n]}_{F}(T)$ which contains $\chi$ and is realized in $M$ has a root in $M$.

**Remark 5.4.** We note that the property of $\chi$-rootedness is expressible by a sentence of $\mathcal{L}_{\omega_1, \omega}$, which asserts that for every tuple $\bar{a}$ of distinct elements satisfying $\chi(\bar{a})$, there is some element $a_i$ of the tuple such that every other tuple $\bar{b}$ with the same $F$-type as $\bar{a}$ contains $a_i$. Hence the set of $\chi$-rooted models of $T$ is a Borel set in $\text{Str}_L$.

Let $M$ be a $\chi$-rooted model of $T$. Suppose that $p(\bar{x}) \in S^{[n]}_{F}(T)$ contains $\chi(\bar{x})$ and is realized in $M$, and let $a$ be a root of $p$ in $M$. If $M \models p(a, \bar{b})$, then $a$ is the unique element of $M$ satisfying $p(x, \bar{b})$, since if $c \neq a$, then $a$ is not in $\bar{c}b$, so $\bar{c}b$ does not realize $p$. This implies that $M$ has non-trivial group-theoretic definable closure, since every automorphism of $M$ fixing $\bar{b}$ also fixes $a$. Note that $T$ may still have trivial definable closure, since $p$ is an $F$-type and, in general, is not equivalent to a formula in $F$.

We can conclude, however, that if an $F$-theory $T$ with trivial definable closure has a $\chi$-rooted model, then no non-redundant type that contains $\chi$ is isolated. Thus isolated types are not dense in $S^{[n]}_{F}(T)$. By standard facts about model theory in countable fragments of $\mathcal{L}_{\omega_1, \omega}$ (see [KK04]), this implies that $T$ does not have a prime model with respect to $F$-elementary embeddings, and that there are continuum-many types in $S^{[n]}_{F}(T)$ containing $\chi(\bar{x})$.

The next theorem says that this situation is typical of properly ergodic models $\mu$: the theory $\text{Th}_F(\mu)$ has many $\chi$-rooted models for some $\chi(\bar{x})$ and (by Theorem 2.17) trivial dcl. Of course, given any non-redundant $F$-type $p$ containing $\chi(\bar{x})$, we can try to bring $\chi$-rootedness into direct conflict with trivial dcl by moving to a larger countable fragment $F'$ which contains the formula $\theta_p(\bar{x}) := \bigwedge_{\varphi \in P} \varphi(\bar{x})$ isolating $p$. But, as we will see, every non-redundant type containing the formula $\chi(\bar{x})$ has measure 0, so $\text{Th}_{F'}(\mu)$ contains the sentence $\forall \bar{x} \neg \theta_p(\bar{x})$, ruling out troublesome realizations of $p$.

It is also worth noting that the countable fragment $F'$ can only isolate and rule out countably many of the continuum-many types of measure 0 containing $\chi(\bar{x})$. In the example of the kaleidoscope random graph (Example 3.1), we could extend from the first-order fragment to a countable fragment $F$ of $\mathcal{L}_{\omega_1, \omega}$ containing some of the conjunctions $\bigwedge_{n \in A} xR_ny \land \bigwedge_{n \notin A} \neg xR_ny$, for $A \in 2^{[n]}$. Then the theory $\text{Th}_{F'}(\mu)$ is essentially the same as $\text{Th}_{F_0}(\mu)$, but with countably many of the continuum-many quantifier-free 2-types forbidden.

**Theorem 5.5.** Let $\mu$ be a properly ergodic structure, $F$ a countable fragment of $\mathcal{L}_{\omega_1, \omega}$, and $\chi(\bar{x})$ a formula in $F$ such that $\mu(\chi(\bar{x})) > 0$. Suppose that $\mu(p) = 0$
for every $F$-type $p$ containing $\chi(\pi)$. Then $\mu$ assigns measure 1 to the set of $\chi$-rooted models of $\text{Th}_F(\mu)$.

**Proof.** Let $L'$, $T'$, and $Q$ be the language, pithy $\Pi_2$ theory, and countable set of partial quantifier-free types obtained from Theorem 2.22, respectively. By Corollary 2.23, $\mu$ corresponds to an ergodic $L'$-structure $\mu'$, concentrated on those models of $T'$ that omit all the types in $Q$.

For such models, each formula $\varphi(\pi)$ in $F$ is equivalent to the atomic $L'$-formula $R_\varphi(\pi)$, so we have $\mu'(R_\chi(\pi)) > 0$, and for every quantifier-free type $q$ containing $R_\chi(\pi)$, $\mu'(q) = 0$. It suffices to show that for an $L'$-structure $M$ sampled from $\mu'$, almost surely every quantifier-free type $p(\pi) \in S^{[n]}_{\text{qf}}(T')$ that contains $R_\chi(\pi)$ and is realized in $M$ has a root in $M$.

By Theorem 2.25, $\mu'$ has an AHK representation $(f_m)_{m \in \mathbb{N}}$. We adopt the notation of Section 2.4 for the random variables $(\xi_A)_{A \in \mathcal{P}_{\text{fin}}([n])}$. Since $\mu'$ is ergodic, by Theorem 2.26, we can pick the functions $f_m$ so they do not depend on the argument indexed by $\emptyset$.

Let $p \in S^{[n]}_{\text{qf}}(T')$. Given a subset $X \subseteq [n]$, we can separate the inputs to $f_n$, $(x_A)_{A \in \mathcal{P}([n])}$, into those contained in $X$ and those not contained in $X$: $((x_A)_{A \subseteq X}, (x_A)_{A \not\subseteq X})$. Then if we fix values $(r_A)_{A \subseteq X}$ from $[0, 1]$, we say $p$ is **likely given** $(r_A)_{A \subseteq X}$ in position if there is a positive probability that $f_n((r_A)_{A \subseteq X}, (\xi_A)_{A \not\subseteq X}) = p$, when $(\xi_A)_{A \not\subseteq X}$ are uniform i.i.d. random variables with values in $[0, 1]$.

Now for any $0 \leq m \leq n$, and any values $(r_A)_{A \in \mathcal{P}([m])}$, we say $p$ is **likely given** $(r_A)_{A \in \mathcal{P}([m])}$ if there is some injective function $i : [m] \to [n]$ such that $p$ is likely given $(r_{i[A]})_{A \in \mathcal{P}([m])}$ in position. We make some observations about this definition:

- In terms of the random variables $(\xi_A)_{A \in \mathcal{P}_{\text{fin}}([n])}$ that we use to sample from the AHK representation of $\mu'$, for a tuple $\bar{b}$ of length at most $n$, $p$ is likely given $\hat{\xi}_B$ (see Section 2.4 for this notation) if, conditioned on the values of $\hat{\xi}_B$, there is a positive probability that $p$ is realized on some tuple containing all the elements of $\bar{b}$.
- Note that this does not depend on the order in which the set $B = |\bar{b}|$ is enumerated as a tuple. Abusing notation, we say that $p$ is likely given $\hat{\xi}_B$ if $p$ is likely given $\hat{\xi}_B$ for some tuple $\bar{b}$ enumerating $B$.
- If $|B| = n$, then all the inputs of $f_n$ are fixed, and $p$ is likely given $\hat{\xi}_B$ if and only if $p$ is realized on $B$ (meaning that some enumeration of $B$ as a tuple realizes $p$).
- If $B = \emptyset$, then, since $f_n$ does not depend on the argument indexed by $\emptyset$, none of the relevant inputs of $f_n$ are fixed, and $p$ is likely given $\hat{\xi}_B$ if and only if $\mu'(p) > 0$. 


Claim 1: Almost surely, for every $B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$ with $|B| \leq n$, and for every $A \in \mathcal{P}_{\text{fin}}(\mathbb{N})$, if $p$ is likely given $\xi_B$, then there is an extension $B \subseteq C$ with $|C| = n$ and $C \cap A = B \cap A$ such that $p$ is realized on $C$.

Proof of Claim 1. This is clear if $|B| = n$, taking $C = B$. If not, let $(C_i)_{i \in \mathbb{N}}$ be extensions of $B$ of size $n$, such that $C_i \cap C_j = B$ when $i \neq j$ and such that $C_i \cap A = B \cap A$ for all $i$. Let $\mathcal{A}_i$ be the event that $p$ is realized on $C_i$. The $\mathcal{A}_i$ are conditionally independent over $\xi_B$, and each has the same positive probability, so almost surely infinitely many occur.

Claim 2: Almost surely, for every $B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$ with $|B| \leq n$, and for every pair of extensions $B \subseteq C$ and $B \subseteq D$ with $C \cap D = B$ and $|C| = |D| = n$, if $p$ is realized on both $C$ and $D$, then $p$ is likely given $\xi_B$.

Proof of Claim 2. Again, this is clear if $|B| = n$, since then $C = D = B$. If not, then since the quantifier-free types realized on $C$ and on $D$ are conditionally independent over $\xi_B$, the probability that the same probability 0 event (e.g., the realization of a type $p$ which is not likely given $\xi_B$) happens on both $C$ and $D$ is 0.

Claim 3: Almost surely, for all $A$ and $B$ in $\mathcal{P}_{\text{fin}}(\mathbb{N})$ with $|A| \leq n$ and $|B| \leq n$, if $p$ is likely given $\xi_A$ and likely given $\xi_B$, then $p$ is likely given $\xi_{A \cap B}$.

Proof of Claim 3. This follows from the last two claims. Suppose $p$ is likely given $\xi_A$ and likely given $\xi_B$. By Claim 1, almost surely, there is an extension $A \subseteq A'$ with $|A'| = n$ such that $A' \cap B = A \cap B$ and $p$ is realized on $A'$. By Claim 1 again, almost surely, there is an extension $B \subseteq B'$ with $|B'| = n$ such that $B' \cap A' = B \cap A' = A \cap B$ and $p$ is realized on $B'$. But then, by Claim 2, almost surely $p$ is likely given $\xi_{A \cap B}$. 

Let $\mathcal{M}$ be the random structure obtained by sampling from the AHK representation. For any non-redundant quantifier-free type $p$ containing $\bar{R}_\chi(\pi)$ which is realized in $\mathcal{M}$, $p$ is likely given $\xi_\pi$ for all $\bar{a}$ realizing $p$, and by Claim 3, almost surely the sets $\{\|\bar{a}\| \mid p \text{ is likely given } \xi_\pi\}$ are closed under intersection. But since $\mu'(p) = 0$, $p$ is not likely given $\xi_\emptyset$. Hence the intersection of all the realizations of $p$ is almost surely non-empty, i.e., $p$ has a root in $\mathcal{M}$.

6. Constructing properly ergodic structures

In this section, we will use a single $\chi$-rooted model $M$ of an $F$-theory $T$ with trivial dcl to construct a properly ergodic structure concentrated on $T$. The strategy is to build a Borel structure $\mathcal{M}$ equipped with a probability measure $\nu$, via an inverse limit of finite probability spaces. We use $M$ as a guide in the construction to ensure that $\mathcal{M}$ is also $\chi$-rooted. Then our ergodic structure $\mu$
will be obtained by i.i.d. sampling of countably many points from $\mathbb{M}$ according to $\nu$ and taking the induced substructure.

Having built the Borel structure $\mathbb{M}$, we proceed to rescale $\nu$, using a technique from [AFKP15], to obtain not just one but continuum-many distinct properly ergodic structures concentrating on $T$.

Definition 6.1. A Borel structure $\mathbb{M}$ is an $L$-structure whose domain is a standard Borel space such that for every relation symbol $R$ of arity $\text{ar}(R)$ in $L$, the subset $R \subseteq M^{\text{ar}(R)}$ is Borel. A measured structure is a Borel structure $\mathbb{M}$ equipped with an atomless probability measure $\nu$.

Given a measured structure $(\mathbb{M}, \nu)$, there is a canonical measure $\mu_{\mathbb{M}, \nu}$ on $\text{Str}_L$, obtained by sampling a countable $\nu$-i.i.d. sequence (of almost surely distinct points) from $\mathbb{M}$ and taking the induced substructure. Somewhat more formally, $\mu_{\mathbb{M}, \nu}$ is the distribution of a random structure in $\text{Str}_L$ whose atomic diagram on $\mathbb{N}$ is given by that of the random substructure of $\mathbb{M}$ with underlying set $\{a_i \mid i \in \mathbb{N}\}$, where $(a_i)_{i \in \mathbb{N}}$ is a $\nu$-i.i.d. sequence of (almost surely unique) elements in $\mathbb{M}$.

We now describe an AHK representation of the measures $\mu_{\mathbb{M}, \nu}$ in the sense of Section 2.4. Choose a measure-preserving Borel isomorphism $h$ from $[0, 1]$ equipped with the uniform measure to the domain of $\mathbb{M}$ equipped with $\nu$, and define functions $f_n : [0, 1]^{P_{\mathbb{M}}([n])} \to S_q^{[n]}(L)$ by

$$f_n((\xi_A)_{A \subseteq [n]}) = \text{qftp}(h(\xi_{\emptyset}), \ldots, h(\xi_{\{n-1\}})).$$

Informally, these functions ignore the random variables $\xi_A$ when $|A| \neq 1$ and view the $(\xi_{\{a\}})_{a \in \mathbb{N}}$ as independent random variables with distribution $\nu$ taking their values in $\mathbb{M}$. (Formally, we also must choose an arbitrary output for $f_n$ on the measure 0 set where $\xi_{\{a\}} = \xi_{\{b\}}$ for some $a \neq b$.)

Now $(f_n)_{n \in \mathbb{N}}$ is an AHK system, so it induces an invariant measure on $\text{Str}_L$. This measure is clearly the same as $\mu_{\mathbb{M}, \nu}$ described above via sampling of a random substructure. Since the $f_n$ do not depend on the argument indexed by $\emptyset$, the measure $\mu_{\mathbb{M}, \nu}$ is ergodic (Theorem 2.26), which establishes the following lemma.

Lemma 6.2. Given a measured structure $(\mathbb{M}, \nu)$, the measure $\mu_{\mathbb{M}, \nu}$ on $\text{Str}_L$ is an ergodic structure.

In fact, this AHK system is “random-free”. This terminology comes from the world of graphons: a graphon is said to be random-free [Jan13, §10] when it is $\{0, 1\}$-valued almost everywhere. This can be thought of as “having randomness” only at the level of vertices (and not at higher levels — namely edges, in the case of graphs). See also 0–1 valued graphons in [LS10] and the simple arrays of [Kal99]. A graphon is random-free if and only if the corresponding AHK system is random-free in the following sense.
Definition 6.3. An AHK system \((f_n)_{n \in \mathbb{N}}\) is **random-free** if each function \(f_n\) depends only on the singleton variables \(\xi_{\{a\}}\) for \(a \in \mathbb{N}\). An ergodic structure \(\mu\) is **random-free** if it has a random-free AHK representation.

We would like to transfer properties of \(M\) to almost-sure properties of \(\mu_M, \nu\). It is not true in general that \(\mu_M, \nu \models \text{Th}_F(M)\). But the following property will allow us to transfer satisfaction in \(M\) to satisfaction in \(\mu_M, \nu\) for pithy \(\Pi_2\) sentences.

Definition 6.4. Let \((M, \nu)\) be a measured structure, and let \(\varphi\) be a pithy \(\Pi_2\) sentence. We say that \((M, \nu)\) **satisfies \(\varphi\) with strong witnesses** (or **has strong witnesses for \(\varphi\)**) if the following hold.

- If \(\varphi\) is universal, then \(M \models \varphi\).
- If \(\varphi = \forall x \exists y \rho(x, y)\), then for every tuple \(\bar{a}\) from \(M\), the set \(\rho(\bar{a}, M) = \{b \in M \mid \rho(\bar{a}, b)\}\) either contains an element of the tuple \(\bar{a}\) or has positive \(\nu\)-measure.

For a pithy \(\Pi_2\) theory \(T\), we say that \((M, \nu)\) **satisfies \(T\) with strong witnesses** when it satisfies \(\varphi\) with strong witnesses for all \(\varphi \in T\).

Note that if \((M, \nu)\) satisfies \(T\) with strong witnesses, then \(M \models T\).

Lemma 6.5. Let \((M, \nu)\) be a measured structure, and let \(\mu = \mu_{M, \nu}\).

(i) Let \(Q\) be a countable set of partial quantifier-free types. If \(M\) omits all the types in \(Q\), then \(\mu\) almost surely omits all the types in \(Q\).

(ii) Let \(T\) be a pithy \(\Pi_2\) theory. If \((M, \nu)\) satisfies \(T\) with strong witnesses, then \(\mu\) almost surely satisfies \(T\).

(iii) Further, if there is a quantifier-free formula \(\chi(x)\) such that \(M\) is \(\chi\)-rooted with respect to quantifier-free types, then \(\mu\) is properly ergodic.

Proof. (i) If no tuple from \(M\) realizes a quantifier-free type \(q \in Q\), then no tuple from any countable substructure sampled from \(M\) realizes \(q\).

(ii) Every universal sentence \(\forall \bar{x} \psi(\bar{x})\) in \(T\) is almost surely satisfied by \(\mu\), since every tuple \(\bar{a}\) from \(M\) satisfies the quantifier-free formula \(\psi(\bar{x})\).

Next, consider sentences of the form \(\forall \bar{x} \exists y \rho(\bar{x}, y)\). Fix an \(n\)-tuple \(\bar{a}\) from \(\mathbb{N}\), where \(n = |\bar{x}|\). Corresponding to this tuple, we have a random tuple \(\bar{v} := (v_{a_1}, \ldots, v_{a_n})\) sampled from \(M\). By Fubini’s theorem, it suffices to show that for each possible value of this random tuple, there is almost surely some \(b \in \mathbb{N}\) such that \(M \models \rho(\bar{v}, v_b)\).

By strong witnesses, \(\rho(\bar{v}, M)\) either contains an element \(v_{a_i}\) of the tuple \(\bar{v}\) or has positive measure. In the first case, \(v_{a_i}\) serves as our witness. In the second case, since there are infinitely many other independent random elements \((v_b)_{b \in \mathbb{N} \setminus \mathbb{N}_1}\), almost surely infinitely many of them land in the set \(\rho(\bar{v}, M)\).

(iii) By (ii), \(\mu \models T\), and since \(\chi(\bar{x}) \land (\bigwedge_{i \neq j} x_i \neq x_j)\) is consistent with \(T\), \(\mu(\chi(\bar{x})) > 0\). Let \(p\) be any type containing \(\chi(\bar{x})\), and let \(q\) be its restriction to
the first-order quantifier-free formulas. To show that \( \mu(p) = 0 \), it suffices to show that \( \mu(q) = 0 \).

Now since \( M \) is \( \chi \)-rooted with respect to quantifier-free types, \( q \) has a root \( v \) in \( M \). The probability that a tuple sampled from \( M \) satisfies \( q \) is bounded above by the probability that the tuple contains \( v \). This is 0, since \( \nu \) is atomless. Hence, by Theorem 4.7, \( \mu \) is properly ergodic. \( \square \)

Thus, after applying the pithy \( \Pi_2 \) transformation from Section 2.3 to an \( F \)-theory \( T \), we have reduced the problem of constructing a properly ergodic structure almost surely satisfying \( T \) to that of constructing a measured structure with the properties in Lemma 6.5.

**Theorem 6.6.** Let \( F \) be a countable fragment of \( L_{\omega_1,\omega} \), \( T \) a complete \( F \)-theory with trivial dcl, \( \chi(\bar{x}) \) a formula in \( F \), and \( M \) a \( \chi \)-rooted model of \( T \). Then there are continuum-many distinct properly ergodic structures \( \mu \) such that \( \mu \models T \).

**Proof.** We begin by applying Theorem 2.22 to obtain a language \( L' \supseteq L \), a pithy \( \Pi_2 \) theory \( T' \), and a countable set of partial quantifier-free types \( Q \). Let \( M' \) be the natural expansion of \( M \) to an \( L' \)-structure. Then \( M' \) is \( R_\chi \)-rooted, where \( R_\chi(\bar{x}) \) is the atomic \( L' \)-formula corresponding to the \( L \)-formula \( \chi(\bar{x}) \). By Corollary 2.23, it suffices to construct a properly ergodic \( L' \)-structure which almost surely satisfies \( T' \) and omits the types in \( Q \).

**Part 1: The inverse system**

We construct a sequence \( (A_k)_{k \in \mathbb{N}} \) of finite \( L' \)-structures, each of which is identified with a substructure of \( M' \). Given a structure \( A \), we define the structure \( A^* \) to have underlying set \( A \cup \{ \ast \} \), where no new relations hold involving \( \ast \). For each \( k \), we equip the underlying set of each \( A_k^* \) with a discrete probability measure \( \nu_k \) that assigns positive measure to every element, and we fix a finite sublanguage \( L_k \) of \( L' \). Finally, we define connecting maps \( f_k: A_{k+1}^* \to A_k^* \) such that \( f_k(\ast) = \ast \) for all \( k \), which preserve the measures and certain quantifier-free types, as follows:

1. \( \nu_{k+1}(f_k^{-1}[X]) = \nu_k(X) \) for all \( X \subseteq A_k^* \).
2. If \( \bar{a} \) is a tuple of distinct elements from \( A_{k+1}^* \) such that \( f_k(\bar{a}) \) is a tuple of distinct elements of \( A_k \), then \( \text{qftp}_{L_k}(\bar{a}) = \text{qftp}_{L_k}(f_k(\bar{a})) \). Note that we make no requirement if \( f_k \) is not injective on \( ||\bar{a}|| \) or if any element of \( \bar{a} \) is mapped to \( \ast \).

We call a pithy \( \Pi_2 \) sentence in \( T' \) an extension axiom if it is not universal. We enumerate the extension axioms in \( T' \) as \( \langle \varphi_k \rangle_{k \in \mathbb{N}} \) and the types in \( Q \) as \( \langle q_i \rangle_{k \in \mathbb{N}} \) with redundancies, so that each axiom and each type appears infinitely often in its list. We also enumerate the symbols in the language \( L' \) as \( \langle R_k \rangle_{k \in \mathbb{N}} \), without redundancies.

At stage 0, we start with \( A_0 = \emptyset \), the empty substructure of \( M' \). Then \( A_0^* = \{ \ast \} \), and we set \( \nu_0(\{ \ast \}) = 1 \) and \( L_0 = \emptyset \).
At stage \( k+1 \), we are given \( A_k, \nu_k \), and \( L_k \). We define \( A_{k+1}, \nu_{k+1}, L_{k+1} \), and the connecting map \( f_k \) in four steps.

**Step 1:** Splitting the elements of \( A_k \).

Enumerate the elements of \( A_k \) as \( \{a_1, \ldots, a_m\} \). We build intermediate substructures of \( M' \), \( B_i = \{a_1, \ldots, a_m, a'_1, \ldots, a'_j\} \), where each new element \( a'_j \) is a “copy” of \( a_j \) to be defined. We start with \( B_0 = A_k \).

Given \( B_i \), let \( \varphi_{B_i}(x_1, \ldots, x_m, x'_1, \ldots, x'_j) \) be the conjunction of all atomic and negated atomic \( L_k \) formulas holding on \( B_i \), so that \( \varphi_{B_i} \) encodes the quantifier-free \( L_k \)-type of \( B_i \). Now there is an \( L \)-formula \( \psi_{B_i} \) in \( F \) such that \( \psi_{B_i} \) has the same realizations as \( \varphi_{B_i} \) in \( M' \). Since \( T = \text{Th}_F(M) \) has trivial dcl, we can find another realization \( a'_{i+1} \neq a_{i+1} \) of \( \psi_{B_i}(a_1, \ldots, x_{i+1}, \ldots, a_m, a'_1, \ldots, a'_i) \) in \( M' \setminus B_i \). Set \( B_{i+1} = B_i \cup \{a'_{i+1}\} \). We have

\[
(\dagger) \quad \text{qftp}_{L_k}(a_1, \ldots, a_{i+1}, \ldots, a_m, a'_1, \ldots, a'_i) = \text{qftp}_{L_k}(a_1, \ldots, a'_{i+1}, \ldots, a_m, a'_1, \ldots, a'_i).
\]

At the end of Step 1, we have a structure \( B_m = \{a_1, \ldots, a_m, a'_1, \ldots, a'_m\} \).

**Step 2:** Splitting \(*\).

The extension axiom \( \varphi_k \) has the form \( \forall \bar{x} \exists y \rho(\bar{x}, y) \), where \( \bar{x} \) is a tuple of length \( j \) and \( \rho(\bar{x}, y) \) is quantifier-free. Suppose there is a tuple \( \bar{a} \) from \( B_m \) such that \( B_m \models \neg \exists y \rho(\bar{a}, y) \). Then, since \( M' \models \exists y \rho(\bar{a}, y) \), we can choose some witness \( c_\bar{x} \) to the existential quantifier in \( M' \setminus B_m \). Let \( W = \{c_\bar{x} \mid \bar{a} \in B'_m \text{ and } B_m \models \neg \exists y \rho(\bar{a}, y)\} \) be the (finite) set of chosen witnesses. Note that if \( \bar{x} \) is the empty tuple of variables, then \( W \) is either empty or consists of a single witness, depending on whether \( B_m \models \exists y \rho(y) \).

Let \( A_{k+1} = B_m \cup W \) if \( W \) is non-empty, and otherwise let \( A_{k+1} = B_m \cup \{c\} \), where \( c \) is any new element in \( M' \setminus B_m \).

**Step 3:** Defining \( f_k \) and \( \nu_{k+1} \).

Recall that \( f_k \) is to be a map from \( A^*_k \) to \( A^*_k \). We set \( f_k(a_i) = f_k(a'_i) = a_i \) and \( f_k(c) = f_k(\ast) = \ast \) for \( c \in A_{k+1} \setminus B_m \).

We define \( \nu_{k+1} \) by splitting the measure of an element of \( A^*_k \) evenly among its preimages under \( f_k \). So \( \nu_{k+1}(a_i) = \nu_{k+1}(a'_i) = \frac{1}{2} \nu_k(a_i) \), and \( \nu_{k+1}(c) = \nu_{k+1}(\ast) = \frac{1}{N} \nu_k(\ast) \), where \( N = |A^*_k \setminus B_m| \geq 2 \). Note that every element of \( A^*_{k+1} \) has positive measure, by induction.

**Step 4:** Defining \( L_{k+1} \).

We expand the current language \( L_k \) to \( L_{k+1} \) by adding finitely many new symbols from \( L' \).

(a) Add \( R_k \) to \( L_{k+1} \) if it is not already included.
(b) Since \( A_{k+1} \) is a substructure of \( M' \), no tuple from \( A_{k+1} \) realizes \( q_k \). That is, for every tuple \( \bar{a} \) from \( A_{k+1} \), there is some quantifier-free formula \( \varphi_{\pi}(\bar{x}) \in q_k \) such that \( M' \models \neg \varphi_{\pi}(\bar{a}) \). Add the finitely many relation symbols appearing in \( \varphi_{\pi} \) to \( L_{k+1} \).
(c) Let \( n \) be the number of free variables in \( \chi(\bar{x}) \). For every pair of \( n \)-tuples \( \bar{a} \) and \( \bar{b} \) from \( A_{k+1} \) that realize distinct quantifier-free \( L' \)-types in \( M' \), there is some relation symbol \( R_{\pi,\bar{a}} \) that separates their types. Add \( R_{\pi,\bar{a}} \) to \( L_{k+1} \).

This completes stage \( k+1 \) of the construction. Let us check that conditions (1) and (2) above are satisfied by the connecting map \( f_k \).

(1): Since \( \nu_k \) and \( \nu_{k+1} \) are discrete measures on finite spaces, it suffices to check that \( \nu_k(a) = \sum_{b \in f_k^{-1}(\{a\})} \nu_{k+1}(b) \) for every singleton \( a \in A^*_k \). This follows immediately from our definitions of \( f_k \) and \( \nu_{k+1} \).

(2): Let \( \bar{b} \) be a tuple from \( A_{k+1} \). The assumption that \( f_k(\bar{b}) \) is a tuple of distinct elements of \( A_k \) means that every element of \( \bar{b} \) is in \( B_m \) (since the other elements are mapped to \( * \)) and that \( a_i \) and \( a'_i \) are not both in \( \bar{b} \) for any \( i \).

For any function \( \gamma : [m] \rightarrow [2] \), let \( \bar{\pi}^\gamma \) be the \( m \)-tuple which contains \( a_i \) if \( \gamma(i) = 0 \) and \( a'_i \) if \( \gamma(i) = 1 \). Then, expanding \( \bar{b} \) to an \( m \)-tuple of the form \( \bar{\pi}^\gamma \), it suffices to show that \( \text{qftp}_{L_k}(\bar{\pi}) = \text{qftp}_{L_k}(f_k(\bar{\pi})) \). This follows by several applications of instances of the equality (\( \dagger \)) above.

**Part 2: The measured structure**

Let \( \mathbb{X} \) be the inverse limit of the system of sets \( A^*_k \) and surjective connecting maps \( f_k \). For each \( k \), let \( \pi_k \) be the projection map \( \mathbb{X} \rightarrow A_k \cup \{*\} \). Then \( \mathbb{X} \) is a profinite set, so it has a natural topological structure as a Stone space, in which the basic clopen sets are exactly the preimages under the maps \( \pi_k \) of subsets of the sets \( A^*_k \). Note that \( \mathbb{X} \) is separable, so it is a standard Borel space.

Let \( \nu^* \) be the finitely additive measure on the Boolean algebra \( \mathcal{B}^* \) of clopen subsets of \( \mathbb{X} \) defined by \( \nu^*(\pi_k^{-1}[X]) = \nu_k(X) \). This is well defined by condition (1). By the Hahn–Kolmogorov Measure Extension Theorem [Tao11, Theorem 1.7.8], \( \nu^* \) extends to a Borel probability measure \( \nu \) on \( \mathbb{X} \).

Now each element \( a \) of \( A^*_k \) has at least 2 preimages in \( A^*_{k+1} \), each of which have measure at most \( \frac{1}{2} \nu_k(a) \). Hence, by induction, the measure of each element of \( A^*_k \) is at most \( 2^{-k} \). So for all \( x \in \mathbb{X} \), the point \( x \) is contained in a basic clopen set \( \mathcal{X}_k = \pi_k^{-1}(\{\pi_k(\bar{x})\}) \) with \( \nu(\mathcal{X}_k) \leq 2^{-k} \) for all \( k \). This implies that \( \nu(\{x\}) = 0 \) and \( \nu \) is non-atomic.

Note that there is a unique element * of \( \mathbb{X} \) with the property that \( \pi_k(*) = * \) for all \( k \). We define a Borel \( L' \)-structure \( \mathcal{M} \) with domain \( \mathbb{X} \setminus \{*\} \) (which is also a standard Borel space). Since we have only removed a measure 0 set from \( \mathbb{X} \), \( \nu \) restricts to a probability measure on \( \mathcal{M} \), which we also call \( \nu \).

We define the structure on \( \mathcal{M} \) by specifying the quantifier-free type of every tuple of distinct elements from \( \mathcal{M} \). By Step 4 (a), \( \bigcup_{k=0}^{\infty} L = L' \). Given a tuple \( \bar{a} \) of distinct elements from \( \mathcal{M} \) and a quantifier-free formula \( \varphi(\bar{x}) \), we choose \( k \) large enough so that \( L_k \) contains all of the relation symbols appearing in \( \varphi(\bar{x}) \) and so that \( \pi_k(\bar{a}) \) is a tuple of distinct elements from \( A_k \). We set \( \mathcal{M} \models \varphi(\pi_k(\bar{a})) \) if and only if \( A_k \models \varphi(\pi_k(\bar{a})) \). This is well-defined by condition (2).
By Step 4 (a), $\bigcup_{k=0}^{\infty} L_k = L'$. Given a tuple $\overline{a}$ from $\mathcal{M}$ and a symbol $R$ in $L'$, we choose $k$ large enough so that $R \in L_k$ and distinct elements of $\overline{a}$ are mapped by $\pi_k$ to distinct elements of $A_k$. We set $\mathcal{M} \models R(\overline{a})$ if and only if $A_k \models R(\pi_k(\overline{a}))$. This is well-defined by condition (2).

According to this definition, to determine whether a quantifier-free formula $\varphi(\overline{x})$ holds of a tuple $\overline{a}$ with repeated elements, we can remove the redundancies from $\overline{a}$ and replace the corresponding variables in $\overline{x}$. For example, if $a_i = a_j$, we can remove $a_j$ and replace instances of $x_j$ in $\varphi(\overline{x})$ with $x_i$. This is equivalent to choosing $k$ large enough so that distinct elements of $\overline{a}$ are mapped by $\pi_k$ to distinct elements of $A_k$ and checking whether $A_k \models \varphi(\pi_k(\overline{a}))$.

The interpretation of a relation symbol $R$ is then a Borel subset of $\mathcal{M}^{ar(R)}$. Indeed, fixing $k$, the set of tuples $\overline{a}$ such that distinct elements of $\overline{a}$ are mapped by $\pi_k$ to distinct elements of $A_k$ and $\pi_k(\overline{a})$ satisfies $R$ is closed (the finite union of certain boxes intersected with certain diagonals), and the interpretation of $R$ is the countable union (over $k$) of these sets. Hence $\mathcal{M}$ is a Borel structure.

We now verify the conditions of Lemma 6.5 for the measured structure $(\mathcal{M}, \nu)$, the pithy $\Pi_2$ theory $T'$, the quantifier-free types $Q$, and the quantifier-free formula $R_\chi(\overline{x})$.

(i) $\mathcal{M}$ omits all the types in $Q$.

Let $q(\overline{x})$ be a type in $Q$, and let $\overline{x}$ be a tuple from $\mathcal{M}$. Let $k$ be large enough so that $\pi_k(\overline{x})$ is a tuple of distinct elements of $A_k$. Since $q$ appears infinitely many times in our enumeration of $Q$, there is some $l > k$ such that $q = q_l$. Then $\overline{b} := \pi_{l+1}(\overline{a})$ is also a tuple of distinct elements of $A_{l+1}$. In Step 4 (b) of stage $l + 1$ of the construction, we ensured that $L_{l+1}$ includes the relation symbols appearing in a quantifier-free formula $\varphi_{\overline{b}}(\overline{x}) \in q_k$ such that $A_{l+1} \models \neg \varphi_{\overline{b}}(\overline{b})$. Then also $\mathcal{M} \models \neg \varphi_{\overline{b}}(\overline{a})$, and hence $\overline{a}$ does not realize $q$.

(ii) $(\mathcal{M}, \nu)$ satisfies $T'$ with strong witnesses.

Let $\varphi$ be a pithy $\Pi_2$ axiom of $T'$. Then $\varphi$ has the form $\forall \overline{x} \psi(\overline{x})$, where $\psi(\overline{x})$ is quantifier-free or has a single existential quantifier. Let $\overline{a}$ be a tuple from $\mathcal{M}$. Let $k$ be large enough so that all the symbols in $\varphi$ are in $L_k$ and $\pi_k(\overline{a})$ is a tuple of disjoint elements of $A_k$.

If $\psi(\overline{x})$ is quantifier-free, then $\mathcal{M} \models \psi(\overline{a})$ if and only if $A_k \models \psi(\pi_k(\overline{a}))$. The latter holds, since $A_k$ is a substructure of $\mathcal{M}$, and $\mathcal{M} \models \varphi$.

Otherwise, $\psi(\overline{x})$ has the form $\exists y \rho(\overline{x}, y)$, and since $\varphi$ appears infinitely many times in our enumeration of the extension axioms in $T'$, there is some $l > k$ such that $\varphi = \varphi_l$. Then $\pi_l(\overline{a})$ is a tuple of distinct elements of $A_l$, and $\overline{b} := \pi_{l+1}(\overline{a})$ is a tuple of distinct elements of $A_{l+1}$. In Step 2 of stage $l + 1$ of the construction, we ensured that there was some witness $c_{\overline{b}}$ such that $A_{l+1} \models \rho(\overline{b}, c_{\overline{b}})$. If $c_\overline{b}$ is not an element of the tuple $\overline{b}$, then for any $c \in \mathcal{M}$ such that $\pi_{l+1}(c) = c_\overline{b}$, we have $\mathcal{M} \models \rho(\overline{a}, c)$. Since $\nu(\pi_{l+1}^{-1}\{c\}) = \nu_{l+1}(c) > 0$, the set $\rho(\overline{a}, \mathcal{M})$ has positive
\(\nu\)-measure. On the other hand, if \(c_\bar{b}\) is an element of the tuple \(\bar{b}\), say \(b_i\), then \(\mathbb{M} \models \rho(\bar{a}, a_i)\).

(iii) \(\mathbb{M}\) is \(R_\chi\)-rooted with respect to quantifier-free types.

We would like to show that every non-redundant quantifier-free \(n\)-type containing \(R_\chi(\bar{x})\) that is realized in \(\mathbb{M}\) has a root in \(\mathbb{M}\). Suppose not. Then there is a quantifier-free type \(\rho(\bar{x})\) and a family of \(n\)-tuples \((\bar{\pi})_{i \in I}\) from \(\mathbb{M}\) such that each \(\bar{\pi}\) realizes \(\rho\), but there is no element \(a\) which is in every \(\bar{\pi}\). Note that if such a family exists, then we can find one containing only finitely many tuples: picking some \(\bar{\pi}\) in the family, for each element \(a_j\) in \(\bar{\pi}\) there is another tuple in the family which does not contain \(a_j\), so \(n + 1\) tuples suffice.

Let \((\bar{\pi}^1, \ldots, \bar{\pi}^m)\) be our finite family of tuples. Let \(k\) be large enough so that \(R_\chi \in L_k\) and \(\pi_k\) is injective on \(\bigcup_{i=1}^{m} [\bar{\pi}^i]\). For all \(i\), let \(\bar{b}^i = \pi_k(\bar{\pi}^i)\). Then all of the tuples \(\bar{b}^i\) realize the same quantifier-free \(L_k\)-type \(p' = p \upharpoonright L_k\) in \(A_k\), and \(p'\) contains \(R_\chi(\bar{x})\). By Step 4 (c) of stage \(k\) of our construction, the tuples \(\bar{b}^i\) must actually realize the same quantifier-free \(L'\) type \(q \supseteq p'\) in \(\mathbb{M}'\) (which may be distinct from \(p\)). But there is no element which appears in all of these tuples, contradicting the fact that \(\mathbb{M}'\) is \(R_\chi\)-rooted.

Let \(\mu = \mu_{\mathbb{M}, \nu}\). By Lemma 6.5, \(\mu\) is a properly ergodic structure that almost surely satisfies \(T'\) and omits the types in \(Q\).

Part 3: Rescaling to obtain continuum-many properly ergodic structures

Again, let \(n\) be the number of free variables in \(\chi(\bar{x})\). For any quantifier-free formula \(\varphi(x_1, \ldots, x_n)\), we define the quantifier-free formula \(\varphi^*(x_1, \ldots, x_n)\):

\[
\bigvee_{\sigma \in S_n} \varphi(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Note that the set \(\varphi^*(\mathbb{M}) = \{\bar{a} \in \mathbb{M}^n | \mathbb{M} \models \varphi^*(\bar{a})\}\) is invariant under the natural action of the symmetric group \(S_n\) on \(\mathbb{M}^n\) by permuting coordinates.

We will use the following claim to apply the rescaling technique from [AFKP15].

**Claim:** There is some quantifier-free formula \(\varphi(\bar{x})\) such that \(0 < \nu^\varphi(\varphi^*(\mathbb{M})) < 1\).

**Proof of Claim.** Suppose not. Then for every quantifier-free formula \(\varphi(\bar{x})\), \(\nu^\varphi(\varphi^*(\mathbb{M}))\) is equal to 0 or 1. Note that \(\nu(\varphi^*(\mathbb{M})) > 0\) if and only if \(\nu(\varphi(\mathbb{M})) > 0\). In particular, since \(\nu(\varphi(\mathbb{M})) > 0\), we have \(\nu^\varphi(\varphi^*(\mathbb{M})) = 1\).

Let \(A \subseteq \mathbb{M}^n\) be the set of tuples satisfying the partial quantifier-free type \(\{\varphi^*(\bar{x}) | \nu^\varphi(\varphi^*(\mathbb{M})) = 1\}\). Since \(A\) is a countable intersection of measure 1 sets, it has measure 1. Pick a tuple \(\bar{a} \in A\). Some permutation of \(\bar{a}\) satisfies \(R_\chi\), and \(A\) is \(S_n\)-invariant, so we may assume that \(\mathbb{M} \models R_\chi(\bar{a})\). Let \(p(\bar{x}) = \text{qftp}(\bar{a})\). Since \(R_\chi(\bar{x}) \in p(\bar{x})\), some coordinate \(a_i\) of \(\bar{a}\) is a root for \(p(\bar{x})\).

Now \(\nu\) is atomless, so the set of tuples in \(\mathbb{M}^n\) containing \(a_i\) has measure 0. Thus we can pick another tuple \(\bar{b} \in A\) which does not contain \(a_i\). By rootedness, no permutation of \(\bar{b}\) satisfies \(p(\bar{x})\).
In particular, there is some quantifier-free formula $\psi(\overline{x}) \in p(\overline{x})$ such that no permutation of $\overline{b}$ satisfies $\psi(\overline{x})$ (explicitly, take the conjunction of $n!$ formulas in $p(\overline{x})$, one separating $p(\overline{x})$ from $\text{qftp}(\sigma(\overline{b}))$ for each $\sigma \in S_n$). But then we have $M \models \psi^*(\overline{a})$ (so $\nu^*(\psi^*(\overline{x})) = 1$), and $M \models \neg \psi^*(\overline{b})$, contradicting the fact that $\overline{b} \in A$. □

In [AFKP15], a method is described for rescaling a probability measure $\mu$ according to a weight $W$ (essentially an assignment of weights to the pieces of a finite partition of the domain) to obtain a new probability measure $\mu^W$. In that paper, all probability measures are continuous measures on $\mathbb{R}$, but the results apply equally well to measures on $M$, since this is a standard Borel space.

The main observation about this construction is that $\mu$ and $\mu^W$ are equivalent measures, in the sense that they are absolutely continuous with respect to each other. It follows that for our measure $\nu$ on $M$, any measure of the form $\nu^W$ is an atomless probability measure on $M$, with the property that $(M, \nu^W)$ satisfies $T'$ with strong witnesses, and hence the measure $\mu^W = \mu_{M, \nu^W}$ on $\text{Str}_L'$ is a properly ergodic structure which almost surely satisfies $T'$ and omits the types in $Q$.

Now by the key proposition [AFKP15, Proposition 3.8], since the set $\varphi^*(M)$ from the Claim above is an $S_n$-invariant Borel set with $\nu^*$-measure strictly between 0 and 1, $\nu^W(\varphi^*(M))$ takes on continuum-many values as $W$ varies through the possible weights. And since, for any tuple $\overline{a}$ of distinct elements of $\mathbb{N}$, $\mu^W([\varphi^*(\overline{a})]) = \nu^W(\varphi^*(\overline{a}))$, this construction produces continuum-many distinct properly ergodic structures of the form $\mu^W$. □

Theorem 6.6, along with the results of the previous sections, gives a “measure-free” characterization of those theories which admit properly ergodic models.

**Theorem 6.7.** Suppose $\Sigma$ is a set of sentences in some countable fragment $F$ of $\mathcal{L}_{\omega_1 \omega}$. The following are equivalent:

1. There is a properly ergodic structure $\mu$ such that $\mu \models \Sigma$.
2. There are continuum-many distinct properly ergodic structures $\mu$ such that $\mu \models \Sigma$.
3. There is a countable fragment $F' \supseteq F$ of $\mathcal{L}_{\omega_1 \omega}$, a complete $F'$-theory $T \supseteq \Sigma$ with trivial dcl, a formula $\chi(\overline{x})$ in $F'$, and a model $M \models T$ which is $\chi$-rooted.

**Proof.** (3) → (2): By Theorem 6.6, there are continuum-many distinct ergodic structures $\mu$ such that $\mu \models T$, and $\Sigma \subseteq T$.

(2) → (1): Clear.

(1) → (3): Theorem 4.7 gives us a countable fragment $F' \supseteq F$, and a formula $\chi(\overline{x})$ in $F'$ such that $\mu(\chi(\overline{x})) > 0$, but for every $F'$-type $p$ containing $\chi(\overline{x})$, $\mu(p) = 0$. Let $T = \text{Th}_F(\mu)$. Then $\Sigma \subseteq T$, and $T$ has trivial dcl by Theorem 2.17. Now by Theorem 5.5, the set of $\chi$-rooted models of $T$ has measure 1. In particular, it is non-empty. □
Remark 6.8. The conditions in Theorem 6.7 (3) can sometimes be satisfied with $F' = F$. In fact, for many of the examples in Section 3, we could take $F'$ to be first-order logic. However, Example 3.6 shows that, in general, the move to a larger fragment of $\mathcal{L}_{\omega_1, \omega}$ is necessary.

The following corollaries, which may be of interest independently of Theorem 6.7, follow immediately from its proof in the case that $\mu$ is properly ergodic and from the analogous construction in [AFP16a] in the case that $\mu$ is almost surely isomorphic to a countable structure.

**Corollary 6.9.** If $\mu$ is an ergodic structure, then for any countable fragment $F$ of $\mathcal{L}_{\omega_1, \omega}$, the theory $\text{Th}_F(\mu)$ has a Borel model (of cardinality $2^{\aleph_0}$).

**Corollary 6.10.** For every countable fragment $F$ of $\mathcal{L}_{\omega_1, \omega}$, every ergodic structure $\mu$ is $F$-elementarily equivalent to a random-free ergodic structure $\mu'$. That is, there exists a random-free ergodic structure $\mu'$ such that $\text{Th}_F(\mu) = \text{Th}_F(\mu')$.

Moreover, except in the case that $\mu$ concentrates on the isomorphism type of a highly homogeneous structure $M$, there exist continuum-many such $\mu'$.

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