Presentating finitary functors

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In the paper On Finitary Functors and their Presentations [1], Adámek, Milius, and Moss prove that every finitary endofunctor on a locally finitely presentable category has a presentation by a signature and a set of flat equations. They give a rather abstract proof of this theorem, appealing in particular to Beck’s Monadicity Theorem. In understanding this theorem, I found it useful to work out a more elementary proof, which I will present in this note.

We begin with the preliminaries and an explanation of what the statement means. See [1] for more details and examples.

A functor $F$ is finitary if it preserves filtered colimits. An object $A$ in a category $C$ is finitely presentable if the functor $\text{Hom}(A,-)$ is finitary. The category $C$ is locally finitely presentable if it is cocomplete and if there is a set $\mathcal{F}$ of finitely presentable objects such that every object in $C$ is a filtered colimit of objects in $\mathcal{F}$. We will view $\mathcal{F}$ as a full subcategory of $C$ and assume that it is skeletal (i.e. it contains exactly one representative from each isomorphism class). I will use lower case letters to denote objects in $\mathcal{F}$ and upper case letters to denote general objects of $C$.

We will use the fact that every object $A$ in $C$ is the filtered colimit of a canonical diagram, consisting of all maps from objects in $\mathcal{F}$ to $A$. Formally, we let $I: \mathcal{F} \to C$ be the inclusion and consider the (small) comma category $(I \downarrow A)$. An object of this category is a map $f: x \to A$ with $x \in \mathcal{F}$, and an arrow from $f: x \to A$ to $g: y \to A$ is a map $h: x \to y$ such that $f = g \circ h$. This category is filtered, and the functor $D: (I \downarrow A) \to C$ given by $D(f: x \to A) = x$ is a diagram in $C$ with colimit $A$. Of course, the co-cone map $D(f: x \to A) \to A$ is $f$ itself. We will abbreviate this by writing

$$A = \lim_{\underset{f: x \to A}{\rightarrow}} x.$$

As a consequence, if $F$ is a finitary functor, then

$$F(A) = \lim_{\underset{f: x \to A}{\rightarrow}} F(x),$$

and the co-cone map on the component indexed by $f$ is $F(f): F(x) \to F(A)$.

A signature $\Sigma$ is collection $(\Sigma_x)_{x \in \mathcal{F}}$ of objects of $D$, indexed by the objects in $\mathcal{F}$. To the signature $\Sigma$, we associate the polynomial functor $H_\Sigma: C \to C$,
defined by

\[ H_\Sigma(A) = \coprod_{f : x \to A} \Sigma_x. \]

If \( \varphi : A \to B \) is an arrow, then \( H_\Sigma(\varphi) : H_\Sigma(A) \to H_\Sigma(B) \) is the map from the coproduct defined on the \( f : x \to A \) component by the inclusion of \( \Sigma_x \) in the \( \varphi \circ f \) component of \( H_\Sigma(B) \):

\[ i_{\varphi \circ f} : \Sigma_x \to \coprod_{g : y \to B} \Sigma_y. \]

So we have \( H_\Sigma(\varphi) \circ i_f = i_{\varphi \circ f} \).

A flat equation in the signature \( \Sigma \) is a parallel pair of maps \( \rho, \rho' : x \to H_\Sigma(y) \). Given a set \( T \) of flat equations, let \( T_{x,y} \) be the set of those pairs \( (\rho, \rho') \in T \) with domain \( x \) and codomain \( H_\Sigma(y) \). Then we define a new signature \( \Sigma_T \) by

\[ (\Sigma_T)_y = \coprod_{(\rho, \rho') \in T_{x,y}} x. \]

Now we obtain a parallel pair of natural transformations \( \eta, \eta' : H_{\Sigma_T} \to H_{\Sigma} \). The component \( \eta_A \) is a map \( H_{\Sigma_T}(A) \to H_{\Sigma}(A) \):

\[ \coprod_{f : y \to A} \coprod_{(\rho, \rho') \in T_{x,y}} x \to H_{\Sigma}(A) \]

which we define on the component of the coproduct indexed by \( f \) and \( (\rho, \rho') \) as the composition

\[ x \xrightarrow{\rho} H_\Sigma(y) \xrightarrow{H_\Sigma(f)} H_\Sigma(A). \]

So we have \( \eta_A \circ i_f \circ i_{(\rho, \rho')} = H_\Sigma(f) \circ \rho. \) We define \( \eta' \) similarly, with \( \rho' \) in place of \( \rho \).

Let’s check naturality of \( \eta \). Given a map \( \varphi : A \to B \), we want to show that the following diagram commutes:

\[ \begin{array}{ccc}
H_{\Sigma_T}(A) & \xrightarrow{\eta_A} & H_{\Sigma}(A) \\
H_{\Sigma_T}(\varphi) \downarrow & & \downarrow H_{\Sigma}(\varphi) \\
H_{\Sigma_T}(B) & \xrightarrow{\eta_B} & H_{\Sigma}(B).
\end{array} \]

Checking commutativity on the component of \( H_{\Sigma_T}(A) \) indexed by \( f : y \to A \) and \( (\rho, \rho') \in T_{x,y} \),

\[ H_\Sigma(\varphi) \circ \eta_A \circ i_f \circ i_{(\rho, \rho')} = H_\Sigma(\varphi) \circ H_\Sigma(f) \circ \rho \]

\[ = H_\Sigma(\varphi \circ f) \circ \rho \]

\[ = \eta_B \circ i_{\varphi \circ f} \circ i_{(\rho, \rho')} \]

\[ = \eta_B \circ H_{\Sigma_T}(\varphi) \circ i_f \circ i_{(\rho, \rho')}. \]

The exact sense in which the pair of natural transformations \( (\eta, \eta') \) encode the equations in \( T \) is captured by the following lemma.
Lemma. Fix objects $A$ and $B$ and a map $\psi: H_\Sigma(A) \to B$. Then $\psi \circ \eta_A = \psi \circ \eta'_A$ if and only if for all flat equations $\rho, \rho': x \to H_\Sigma(y)$ in $T$ and all maps $f: y \to A$, we have $\psi \circ H_\Sigma(f) \circ \rho = \psi \circ H_\Sigma(f) \circ \rho'$.

Proof. The equality $\psi \circ \eta_A = \psi \circ \eta'_A$ holds if and only if $\psi \circ \eta_A \circ i_f \circ i_{(\rho, \rho')} = \psi \circ \eta'_A \circ i_f \circ i_{(\rho, \rho')}$ holds for all $f: y \to A$ and all $(\rho, \rho') \in T_{x,y}$. Reversing the quantifiers and rewriting, this is true if and only if for all $\rho, \rho': x \to H_\Sigma(y)$ in $T$ and all maps $f: y \to A$, we have $\psi \circ H_\Sigma(f) \circ \rho = \psi \circ H_\Sigma(f) \circ \rho'$.

Let $G: C \to C$ be a functor, and let $\varepsilon: H_\Sigma \to G$ be a natural transformation. We say that $G$ is presented by the equations $T$ in the signature $\Sigma$ if $\varepsilon$ is the coequalizer of $\eta$ and $\eta'$ in the category of functors $C \to C$:

$$H_\Sigma \xrightarrow{\eta} H_\Sigma \xrightarrow{\varepsilon} G.$$

It is a fact that all polynomial functors and all functors presented by flat equations are finitary. Our goal is to prove that the converse is true, and we now have enough definitions in place to do so (the reader may skip to the proof of the Theorem below). But first, I will provide some motivation for this notion, observing that if a functor $F$ is presented by the equations $T$ in signature $\Sigma$, then an $F$-algebra is “the same as” an $H_\Sigma$-algebra which satisfies all the equations in $T$.

Let $F: C \to C$ be a functor. An $F$-algebra is an object $A \in C$ together with a map $\alpha: F(A) \to A$.

Let $(A, \alpha)$ be an $H_\Sigma$-algebra. We say that $A$ satisfies the flat equation $\rho, \rho': x \to H_\Sigma(y)$ if for all maps $f: y \to A$, we have $\alpha \circ H_\Sigma(f) \circ \rho = \alpha \circ H_\Sigma(f) \circ \rho'$. Taking $B = A$ and $\psi = \alpha$ in the Lemma, we see that $(A, \alpha)$ satisfies all the equations in $T$ if and only if $\alpha \circ \eta_A = \alpha \circ \eta'_A$.

If $\varepsilon: F \to G$ is a natural transformation, then any $G$-algebra $(A, \alpha)$ also carries the structure of an $F$-algebra, with structure map $\alpha \circ \varepsilon_A: F(A) \to A$.

Claim. If $G$ is presented by the equations $T$ in the signature $\Sigma$, then the above correspondence puts $G$-algebras in bijection with $H_\Sigma$-algebras which satisfy all the equations in $T$.

Proof. Suppose $(A, \alpha)$ is a $G$-algebra. We claim that the $H_\Sigma$-algebra $(A, \alpha \circ \varepsilon_A)$ satisfies all the equations in $T$. As we observed above, it suffices to show that $\alpha \circ \varepsilon_A \circ \eta_A = \alpha \circ \varepsilon_A \circ \eta'_A$. But this is true, since $\varepsilon$ is the coequalizer of $\eta$ and $\eta'$.

Conversely, suppose $(A, \alpha)$ is an $H_\Sigma$-algebra which satisfies all the equations in $T$. Then $\alpha \circ \eta_A = \alpha \circ \eta'_A$. Since $\varepsilon_A$ is the coequalizer of $\eta_A$ and $\eta'_A$, we obtain an arrow $\beta: G(A) \to A$, which makes $A$ into a $G$-algebra:

$$H_\Sigma(A) \xrightarrow{\eta_A} H_\Sigma(A) \xrightarrow{\varepsilon_A} G(A) \xrightarrow{\beta} A,$$

where $\alpha \circ \varepsilon_A = \alpha$, so these correspondences are inverses. \qed
With a little more work, one can show that this bijection gives an isomorphism of categories between the category $G\text{-Alg}$ and the full subcategory of $H\Sigma\text{-Alg}$ consisting of those algebras satisfying all the equations in $T$.

Now let’s prove the main theorem.

**Theorem.** Every finitary functor $F: \mathcal{C} \to \mathcal{C}$ can be presented by a signature and a set of flat equations.

**Proof.** First, we define the signature $\Sigma$. For all $x \in \mathcal{F}$, let $\Sigma_x = F(x)$. Thus

$$H_\Sigma(A) = \coprod_{f : x \to A} F(x).$$

There is a natural transformation $\varepsilon: H_\Sigma \to F$. Its component $\varepsilon_A: H_\Sigma(A) \to F(A)$ is defined on the component of the coproduct indexed by $f : x \to A$ as $F(f): F(x) \to F(A)$.

As an aside, this is a very natural choice of signature. Since $F$ is finitary, for any object $A$, $\varepsilon_A$ is just the obvious map $\coprod_{f : x \to A} F(x) \to \lim_{f : x \to A} F(x) = F(A)$, which is the identity $\text{id}_{F(x)}$ on the component indexed by $f : x \to A$, followed by the co-cone map $F(f)$.

Let’s check that $\varepsilon$ is natural. Given a map $\varphi: A \to B$, we want to show that the following diagram commutes:

$$\begin{array}{ccc}
H_\Sigma(A) & \xrightarrow{\varepsilon_A} & F(A) \\
\downarrow_{H_\Sigma(\varphi)} & & \downarrow_{F(\varphi)} \\
H_\Sigma(B) & \xrightarrow{\varepsilon_B} & F(B).
\end{array}$$

Checking commutativity on the component of $H_\Sigma(A)$ indexed by $f : x \to A$,

$$F(\varphi) \circ \varepsilon_A \circ i_f = F(\varphi) \circ F(f)$$

$$= F(\varphi \circ f)$$

$$= \varepsilon_B \circ i_{\varphi \circ f}$$

$$= \varepsilon_B \circ H_\Sigma(\varphi) \circ i_f.$$

Let $T = \{\rho, \rho': x \to H_\Sigma(y) \mid \varepsilon_y \circ \rho = \varepsilon_y \circ \rho'\}$. As above, the set $T$ of flat equations induces natural transformations $\eta, \eta': H_\Sigma_T \to H_\Sigma$. Our claim is that $\varepsilon$ is the coequalizer of $\eta$ and $\eta'$.

First, we show that $\varepsilon \circ \eta = \varepsilon \circ \eta'$. Fix an object $A$, and consider the component of $H_\Sigma_T(A)$ indexed by $f : y \to A$ and $(\rho, \rho') \in T_{x,y}$. Since $\varepsilon_A \circ \eta_A \circ i_f \circ i_{(\rho, \rho')} = \varepsilon_A \circ H_\Sigma(f) \circ \rho$ and similarly for $\eta'_A$, we would like to show that
\( \varepsilon_A \circ H_{\Sigma}(f) \circ \rho = \varepsilon_A \circ H_{\Sigma}(f) \circ \rho' \). This follows from naturality of \( \varepsilon \), and the fact that \( \varepsilon_y \circ \rho = \varepsilon_y \circ \rho' \):

\[
\begin{array}{ccc}
x & \xrightarrow{\rho} & H_{\Sigma}(y) \xrightarrow{H_{\Sigma}(f)} H_{\Sigma}(A) \\
\varepsilon_y & \Downarrow & \varepsilon_A \\
F(y) & \xrightarrow{F(f)} & F(A).
\end{array}
\]

Next, suppose there is another functor \( G \) and a natural transformation \( \varepsilon^*: H_{\Sigma} \rightarrow G \) such that \( \varepsilon^* \circ \eta = \varepsilon^* \circ \eta' \). We must show that there is a unique natural transformation \( \zeta: F \rightarrow G \) making the diagram commute:

\[
\begin{array}{ccc}
H_{\Sigma_{\tau}} & \xrightarrow{\eta} & H_{\Sigma} \\
\varepsilon^* & \downarrow & \varepsilon \\
F(\eta) & \xrightarrow{1_{\zeta}} & F \xrightarrow{1_{\zeta'}} G.
\end{array}
\]

Fix an object \( A \). To define \( \zeta_A: F(A) \rightarrow G(A) \), it suffices to give a coherent family of maps \( \zeta^ f_A: F(x) \rightarrow G(A) \), since

\[
F(A) = \lim_{f: x \rightarrow A} F(x).
\]

Then the triangle that we want to commute can be rewritten as

\[
\begin{array}{ccc}
\prod_{f: x \rightarrow A} F(x) & \xrightarrow{\varepsilon_A} & \lim_{f: x \rightarrow A} F(x) \\
\varepsilon_A & \downarrow & \varepsilon_A \\
G(A) & \xrightarrow{i_{\zeta}} & G(A)
\end{array}
\]

Recalling that \( \varepsilon_A \) maps the component of the coproduct indexed by \( f \) to the component of the colimit indexed by \( f \) by the identity, we are forced to define \( \zeta^ f_A = \varepsilon^*_A \circ i_f \) (where, as usual, \( i_f \) is the inclusion of \( F(x) \) in the component of the coproduct indexed by \( f \)). The fact that this choice was forced on us establishes uniqueness.

It remains to check that the maps \( \zeta^ f_A \) cohere to a map \( F(A) \rightarrow G(A) \). That is, given a commutative triangle

\[
\begin{array}{ccc}
x & \xrightarrow{f} & A \\
\downarrow h & & \downarrow \circ \zeta^ f_A \\
y & \xrightarrow{g} & F(A)
\end{array}
\]

we must show that \( \zeta^ f_A = \zeta^ g_A \circ F(h) \). Equivalently, that \( \varepsilon^*_A \circ i_f = \varepsilon^*_A \circ i_g \circ F(h) \).
For reference, here is a diagram containing all of the arrows which will be used in the proof:

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\[ \begin{array}{ccc}
  z & \xrightarrow{j} & F(x) \\
  \downarrow & & \downarrow \\
  F(h) & \rightarrow & H_{\Sigma}(w) \\
  \uparrow & & \uparrow \\
  F(y) & \rightarrow & H_{\Sigma}(A) \\
  \downarrow & & \downarrow \\
  F(w) & \rightarrow & G(A)
\end{array} \]
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We now use the fact that \( C \) is locally finitely presentable in two key ways. Since \( F(x) \) is a filtered colimit of finitely presentable objects, it suffices to show that for all \( j: z \rightarrow F(x) \),

\[ \varepsilon_w \circ \rho = \varepsilon_w \circ \rho' \]

Since \( x \) and \( y \) are finitely presentable and \( A \) is a filtered colimit of finitely presentable objects, the maps \( f: x \rightarrow A \) and \( g: y \rightarrow A \) factor through some map \( k: w \rightarrow A \) as in the following diagram:

```
\[ \begin{array}{ccc}
  x & \xrightarrow{f} & A \\
  \downarrow & & \downarrow \\
  h & \rightarrow & k \\
  \uparrow & & \uparrow \\
  y & \rightarrow & \epsilon_w \circ i_f \circ j
\end{array} \]
```

Now consider the flat equation \( \rho, \rho': z \rightarrow H_{\Sigma}(w) \), where we define \( \rho = i_f \circ j \) and \( \rho' = i_\rho \circ F(h) \circ j \). I claim that \( (\rho, \rho') \in T \), i.e. that \( \varepsilon_w \circ \rho = \varepsilon_w \circ \rho' \). Indeed,

\[ \varepsilon_w \circ \rho' = \varepsilon_w \circ i_\rho \circ F(h) \circ j \\
    = F(g') \circ F(h) \circ j \\
    = F(f') \circ j \\
    = \varepsilon_w \circ i_f \circ j \\
    = \varepsilon_w \circ \rho. \]

Now by the Lemma, applied to the flat equation \( \rho, \rho': z \rightarrow H_{\Sigma}(w) \) and the map \( k: w \rightarrow A \), since \( \varepsilon_A \circ \eta_A = \varepsilon_A \circ \eta_A' \), we have \( \varepsilon_A \circ H_{\Sigma}(k) \circ \rho = \varepsilon_A \circ H_{\Sigma}(k) \circ \rho' \). Rewriting,

\[ \varepsilon_A \circ H_{\Sigma}(k) \circ i_f \circ j = \varepsilon_A \circ H_{\Sigma}(k) \circ i_\rho \circ F(h) \circ j \]
\[ \varepsilon_A \circ H_{\Sigma}(k) \circ i_{k'f} \circ j = \varepsilon_A \circ i_{k'g'} \circ F(h) \circ j \]
\[ \varepsilon_A \circ i_f \circ j = \varepsilon_A \circ i_g \circ F(h) \circ j \]

as was to be shown. \( \square \)
References