Foundations of Cologic

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Motivation 1: Projective Fraïssé theory

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![Diagram](image)

(arrows are embeddings)
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- Under these conditions, $K$ has a Fraïssé limit $M$, constructed as a direct limit of finite structures in $K$ along embeddings, and satisfying universality and homogeneity. Going back and forth, we also have ultrahomogeneity: any two isomorphic substructures are conjugate by an automorphism.
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The first-order theories of Fraïssé limits are exactly the countably categorical theories with quantifier elimination. The “embedding extension properties” are expressible by $\forall \exists$ axioms.
Motivation 1: Projective Fraïssé theory

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Now let $K$ a class of finite structures (in a finite relational language), considered together with certain surjective maps ("coembeddings") between them.
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Now let \( K \) a class of finite structures (in a finite relational language), considered together with certain surjective maps ("coembeddings") between them.

\( K \) is a projective Fraïssé class if it is closed under quotients by coembeddings (dual to HP), and satisfies the duals of JEP and AP.
Motivation 1: Projective Fraïssé theory

Under these hypotheses, $K$ has a projective Fraïssé limit $M$, constructed as an inverse limit of finite structures in $K$ along coembeddings, and satisfying the duals of universality and homogeneity.

Here the maps out of $M$ are continuous coembeddings:

- As an inverse limit of finite sets, $M$ is naturally a Stone space (compact, Hausdorff, zero-dimensional = basis of clopen sets).
- We equip the finite structures in $K$ with the discrete topology.
Motivation 1: Projective Fraïssé theory

Going back-and-forth, a projective Fraïssé limit $M$ is ultracohomogeneous: Any two isomorphic finite quotients are conjugate by an isomorphism.

\[ M \xrightarrow{\sim} A \]

\[ M \xrightarrow{\sim} A' \]

Ideas:

- There should be a full first-order "cologic", in which:
  - Finite quotients play the role of finite substructures.
  - The "coembedding extension properties" are expressible as $\forall \exists$ axioms.

Projective Fraïssé theory is the special case of "cocountably categorical" cotheories with quantifier elimination.
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- Finite quotients play the role of finite substructures.
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Corelations and costructures

For ordinary first-order structures:
An \( n \)-tuple from \( M \) is a map \([n] \to M \) (\( [n] = \{1, \ldots, n\} \)).
We denote the set of \( n \)-tuples by \( M^n \).
An \( n \)-ary relation \( R \) on \( M \) is a set of \( n \)-tuples from \( M \), \( R \subseteq M^n \).
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**Definition (Solecki)**

An $n$-cotuple from a Stone space $M$ is a continuous map $M \rightarrow [n]$.
We denote the set of $n$-cotuples by $[n]^M$.
An $n$-ary corelation $R$ on $M$ is a set of $n$-cotuples from $M$: $R \subseteq [n]^M$. 
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**Definition**

A *core relational signature* is a set of corelation symbols \( \mathcal{R} \), together with an
arity \( \text{ar}(R) \geq 1 \) for each \( R \in \mathcal{R} \).
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**Definition**

A *corelational signature* is a set of corelation symbols $\mathcal{R}$, together with an arity $\text{ar}(R) \geq 1$ for each $R \in \mathcal{R}$.

**Definition**

A *costructure* for the corelational cosignature $\mathcal{R}$ is a Stone space $M$ together with an $\text{ar}(R)$-ary corelation $R^M$ on $M$ for each $R \in \mathcal{R}$.
We sometimes think of an $n$-cotuple $f : M \to [n]$ as a labeled partition of $M$ into $n$ clopen sets, $f^{-1}({1}), \ldots, f^{-1}({n})$.

Note that some of the pieces may be empty, if $f$ is not surjective.
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Corelations (and hence coformulas) express properties of partitions.
Coembeddings

Let $M$ and $N$ be costructures. Given a continuous map $f : M \to N$, any $n$-cotuple $A : N \to \mathbb{n}$ pulls back to an $n$-cotuple $A \circ f : M \to \mathbb{n}$.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
& \downarrow & \downarrow \\
& A & \xrightarrow{} \mathbb{n}
\end{array}
\]

**Definition**

A continuous map $f : M \to N$ is a *coembedding* if

1. It is surjective, and
2. $A \in R^N$ if and only if $(A \circ f) \in R^M$, for every $n$-ary corelation $R$ in the language and every $n$-cotuple $A$ from $N$. 
Why topology?

Set $\cong \text{ind-FinSet}$: Every set is the filtered colimit of its finite subsets.
Stone $\cong \text{pro-FinSet}$: Every Stone space is the cofiltered limit of its discrete finite quotients.

The topology on $S$ captures the pro-structure:
A basic clopen set in $S$ is the preimage of a subset of $A_i$ for some $i$.
A map $T \to S$ is continuous iff it is induced by a coherent family of maps between the finite quotients of $T$ and $S$.

Slogan: Logic explores infinite structures via their finite subsets. Cologic explores infinite costructures via their finite quotients.
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\[
\begin{aligned}
  S 
  \rightarrow & \ A_3 & \rightarrow & A_2 & \rightarrow & A_1 \\
  \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
  \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
  \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
  \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{aligned}
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A final word on projective Fraïssé theory

It appears that corelational signatures provide the the “correct” general context for projective Fraïssé theory.

**Theorem (Panagiotopoulos, ’16)**

1. Let $S$ be any second-countable Stone space. Then a subgroup $G$ of $\text{Homeo}(S)$ is closed in the compact-open topology if and only if $G = \text{Aut}(M)$ for some projective Fraïssé limit $M$ with domain $S$ in a corelational signature.

2. Let $Y$ be any second-countable compact Hausdorff space. Then there is a projective Fraïssé class $K$ in a corelational signature, such that the projective Fraïssé limit $M$ admits a canonical equivalence relation $\sim$, and $M/\sim$ is homeomorphic to $Y$.

This generalizes many previous examples in which compact Hausdorff spaces were realized as quotients of projective Fraïssé limits.
Let $X = \{x_1, \ldots, x_k\}$ be a finite set of “covariables”. An $n$-cotuple of covariables in context $X$ is a map $t: X \to \mathbb{[n]}$.

We can represent an $n$-cotuple by an $n$-tuple describing a partition of $X$.

**Example:** $(x_1 \sqcup x_3, \emptyset, x_2)$ is a 3-cotuple in context $X = \{x_1, x_2, x_3\}$.

Think of the covariables as labeling a clopen partition of a costructure $M$. 
An atomic coformula in context $X$ is:

- $R(t(X))$, where $t$ is an $ar(R)$-cotuple of covariables in context $X$.
- $\Box_i t(X)$, where $t$ is an $n$-cotuple of covariables in context $X$ and $1 \leq i \leq n$. (This is the dual of equality.)

A coformula in context $X$ is:

- An atomic coformula in context $X$.
- A Boolean combination of coformulas in context $X$.
- $\exists(y \sqcup z = x_i) \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$, or $\forall(y \sqcup z = x_i) \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$, where $\psi$ is a coformula in context $(X \setminus \{x_i\}) \cup \{y, z\}$
Semantics

Let $M$ be a costructure given together with an $X$-cotuple $A : M \to X$, and let $\varphi(X)$ be a coformula in variable context $X$. We will define the satisfaction relation $M \models \varphi(A)$.

Any $n$-cotuple of covariables in context $X$, $t : X \to [n]$, induces an $n$-cotuple $t \circ A : M \to [n]$ from $M$ by composition, which we denote $t(A)$.

\[
\begin{array}{ccc}
M & \xrightarrow{A} & X \\
\mathbb{1} & \xrightarrow{t(A)} & [n]
\end{array}
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\[ M \xrightarrow{A} X \xrightarrow{t} [n] \]

- $M \models R(t(A))$ iff $t(A) \in R^M$.
- $M \models \Downarrow_i t(A)$ iff $t(A)^{-1}(\{i\}) = \emptyset$.
- The usual satisfaction rules hold for Boolean combinations.

Note: In ordinary logic, equality tests for injectivity of a tuple $[n] \to M$. In cologic, coequality ($\Downarrow_i$) tests for surjectivity of a cotuple $M \to [n]$. 
Let $X = \{x_1, \ldots, x_n\}$, and let $\hat{X} = (X \setminus \{x_i\}) \cup \{y, z\}$.
Define $s_i: \hat{X} \to X$ by $s_i(x_j) = x_j$ for $j \neq i$ and $s_i(y) = s_i(z) = x_i$.
A lift of the $X$-cotuple $A: M \to X$ is an $\hat{X}$-cotuple $\hat{A}: M \to \hat{X}$ such that $s_i \circ \hat{A} = A$. 

\[
\begin{array}{ccc}
M & \xrightarrow{\hat{A}} & \hat{X} \\
\downarrow{A} & & \downarrow{s_i} \\
X & & X
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Semantics for quantifiers

Let $X = \{x_1, \ldots, x_n\}$, and let $\hat{X} = (X \setminus \{x_i\}) \cup \{y, z\}$. Define $s_i: \hat{X} \to X$ by $s_i(x_j) = x_j$ for $j \neq i$ and $s_i(y) = s_i(z) = x_i$. A lift of the $X$-cotuple $A: M \to X$ is an $\hat{X}$-cotuple $\hat{A}: M \to \hat{X}$ such that $s_i \circ \hat{A} = A$.

\[
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M & \xrightarrow{\hat{A}} & \hat{X} \\
\downarrow A & \downarrow s_i & \downarrow X
\end{array}
\]

- $M \models \exists(y \sqcup z = x_i) \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$ iff there exists a lift $\hat{A}: M \to \hat{X}$, such that $M \models \psi(\hat{A})$.
- $M \models \forall(y \sqcup z = x_i) \psi(\{x_1, \ldots, y, z, \ldots, x_n\})$ iff for all lifts $\hat{A}: M \to \hat{X}$, $M \models \psi(\hat{A})$.

Note: A lift $\hat{A}$ of $A$ is a refinement of the partition given by $A$. So we are quantifying over finer partitions of $M$. 

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Cosentences and cotheories

**Definition**

A cosentence is a coformula in the singleton context $\ast$.

**Note:** Any costructure $M$ has a unique $\ast$-cotuple $!: M \to \ast$. 

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We will suppress the singleton context for sentences:

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- We will write $M \models \varphi$ instead of $M \models \varphi(\!)$. 


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A cotheory is a set of cosentences.
Example: What can you say in the empty language?

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There exists an isolated point:

\[ \exists (x_1 \sqcup x_2 = *) \left( \neg \Box_1 (x_1, x_2) \land \forall (y_1 \sqcup y_2 = x_1) \right) \]
\[ (\Box_1 (y_1, y_2, x_2) \lor \Box_2 (y_1, y_2, x_2)) \]
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There exists a clopen perfect set:

\[ \exists (x_1 \sqcup x_2 = *) (\neg \Box_1 (x_1, x_2) \land \forall (y_1 \sqcup y_2 = x_1) (\neg \Box_1(y_1, y_2, x_2) \rightarrow \\
\exists (z_1 \sqcup z_2 = y_1) (\neg \Box_1(z_1, z_2, y_2, x_2) \land \neg \Box_2(z_1, z_2, y_2, x_2))) \]
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\[ \textbf{Theorem} \]

\textit{Stone spaces are coelementarily equivalent (i.e. they satisfy the same cosentences in the empty language) if and only if their Boolean algebras of clopen sets are elementarily equivalent.}

Elementary classes of Boolean algebras are classified by Tarski invariants.
[Cherlin, van den Dries, Macintyre, Chatzidakis]

In an influential unpublished paper, “The elementary theory of regularly closed fields”, Cherlin, van den Dries, and Macintyre introduced a “cologic” of profinite groups (e.g. Galois groups) in order to study the model theory of PAC fields. CDM cologic is just ordinary first-order logic on a multisorted structure encoding the full inverse system of finite quotients of a profinite group $G$:

One sort for each $n \geq 1$. Sort $n$ consists of the disjoint union of all finite quotients of $G$ of size $n$.

A ternary relation $\cdot_n$ for each sort $n$, such that $\cdot_n(x,y,z)$ iff all three elements live in the same finite quotient of size $n$, and $x \cdot y = z$.

A binary relation $\pi_{m,n}$ for each pair of sorts $m \geq n$, such that $\pi_{m,n}(x,y)$ iff $x \in H_1$ of size $m$, $y \in H_2$ of size $n$, and the quotient map $\pi_{H_2}:G \to H_2$ factors through the quotient map $\pi_{H_1}:G \to H_1$, as $\pi_{H_2} = \rho \circ \pi_{H_1}$, and $\rho(x) = y$.
Motivation 2: The cologic of profinite groups

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- A ternary relation $\cdot_n$ for each sort $n$, such that $\cdot_n(x, y, z)$ iff all three elements live in the same finite quotient of size $n$, and $x \cdot y = z$.
- A binary relation $\pi_{m,n}$ for each pair of sorts $m \geq n$, such that $\pi_{m,n}(x, y)$ iff $x \in H_1$ of size $m$, $y \in H_2$ of size $n$, and the quotient map $\pi_{H_2} : G \to H_2$ factors through the quotient map $\pi_{H_1} : G \to H_1$, as $\pi_{H_2} = \rho \circ \pi_{H_1}$, and $\rho(x) = y$. 
**Option 1:** Let $L$ be the corelational signature with one $n$-ary corelation symbol $R_H$ for each finite group $H$ with domain $[n]$.

To any profinite group $G$, we can associate a canonical $L$-costructure with the same underlying Stone space $G$:
Let $A : G \to [n]$ be an $n$-cotuple. Then, for any group $H$ with domain $[n]$, $G \models R_H(A)$ if and only if $A : G \to H$ is a surjective group homomorphism.
Profinite groups as costructures

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**Option 2:** Develop cologic for pro-$C$ objects, where $C$ is any category with finite limits.

- Taking $C = \text{FinSet}$ gives cologic in the sense of this talk.
- Taking $C = \text{FinGrp}$ makes profinite groups costructures in the empty language. e.g. For a finite group $H$, an “$H$-cotuple” from a profinite group $G$ is a continuous homomorphism $G \to H$.

This can be done! In fact, it seems like the right level of generality.
Costructures as presheaf structures

Following CDM, we can encode a costructure \( M \) as an ordinary many-sorted first-order structure:

- One sort \( S_n \) for each \( n \geq 0 \), interpreted as \( [n]^M (= \text{Hom}(M, [n])) \).
- One function symbol \( f : S_m \to S_n \) for each function \( f : [m] \to [n] \), interpreted as \( (f \circ -) : [m]^M \to [n]^M \).
- One unary relation symbol \( R \) on sort \( S_n \) for each corelation symbol \( R \), interpreted in the obvious way.

A structure in this language is called a \textit{presheaf structure}, because the sorts and functions encode a functor \( \text{FinSet} \to \text{Set} \) (a presheaf on \( \text{FinSet}^{\text{op}} \)).
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Stone$^{\text{op}} = (\text{pro-}\text{FinSet})^{\text{op}}$ can be identified via $S \mapsto \text{Hom}(S, -)$ with the subcategory of $\text{Set}^{\text{FinSet}}$ consisting of those functors $\text{FinSet} \to \text{Set}$ which preserve finite limits. This gives rise to a duality:

$$(\text{Costructures, Coembeddings})^{\text{op}} \cong (\text{Mod}(T_{\text{lim}}), \text{Embeddings})$$

Where $T_{\text{lim}}$ is the theory in the language of presheaf structures asserting that finite limits are preserved.
The sentences and formulas of cologic correspond to sentences and formulas in a *fragment* of first-order logic over presheaf structures: all formulas are unary, all quantifiers are bounded $[\forall (x \in f^{-1}(y)) \varphi(x)]$, etc. But modulo $T_{\text{lim}}$, this fragment is essentially as expressive as full first-order logic.
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Having “interpreted” cologic in many-sorted first-order logic, we get:

**Corollary (Compactness)**

*A cotheory $T$ is satisfiable if and only if every finite subset of $T$ is satisfiable.*
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Duality is useful for transporting theorems from ordinary first-order logic to cologic. But I also believe that it’s valuable to have a natural syntax for cologic that refers directly to the intended semantics, rather than resorting to duality at every turn.
Motivation 3: Ultracoproducts of compact spaces

[Bankston]
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[Bankston]

Ultraproducts can be described categorically as direct limits of products:

$$\prod_{U} M_i = \prod_{i \in I} M_i / U \cong \lim_{\rightarrow} \prod_{X \in U, i \in X} M_i$$

with connecting maps the projections $$\prod_{i \in X} M_i \rightarrow \prod_{i \in Y} M_i$$ when $$Y \subseteq X$$. 

Motivation 3: Ultracoproducts of compact spaces

[Bankston]

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with connecting maps the projections \( \prod_{i \in X} M_i \to \prod_{i \in Y} M_i \) when \( Y \subseteq X \).

To define the ultracoproduct, just dualize:

\[
\bigsqcup_{i \in X} M_i \cong \lim_{\longleftarrow} \prod_{i \in X} M_i
\]

with connecting maps the inclusions \( \bigsqcup_{i \in Y} M_i \to \bigsqcup_{i \in X} M_i \) when \( Y \subseteq X \).
Motivation 3: Ultracoproducts of compact spaces

\[
\biguplus_{U} M_i \cong \lim_{\leftarrow X \in U} \biguplus_{i \in X} M_i
\]

**Problem:** When the \( M_i \) are sets, \( \biguplus_{i \in X} M_i \) is disjoint union, \( \lim_{\leftarrow} \) is intersection, and when \( U \) is nonprincipal, \( \bigcup_{U} M_i \) is empty!
Motivation 3: Ultracoproduts of compact spaces

\[ \bigsqcup_{i \in \text{X}} M_i \cong \varprojlim_{X \in U} \bigsqcup_{i \in X} M_i \]

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**Bankston’s Solution:** Work in the category of compact Hausdorff spaces. Then the infinite coproduct \( \bigsqcup_{i \in I} M_i \) is the Stone-Čech compactification of the disjoint union, and the intersection is nontrivial.
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Bankston has developed a remarkable amount of dualized model theory for compact Hausdorff spaces, without any syntax.

For example, two spaces \( S \) and \( T \) are defined to be colementarily equivalent if and only if they have homeomorphic ultracopowers!
Ultracoproducts of costructures

Let $\left(M_i\right)_{i \in I}$ be a family of costructures, and let $U$ be an ultrafilter on $I$. Define $\coprod_U M_i = \lim_{\leftarrow X \in U} \bigsqcup_{i \in X} M_i$ (limits and coproducts taken in Stone).
Ultracoproducts of costructures

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The discrete space \([n]\) is *cocompact* in Stone: the functor \(\text{Hom}(−, [n])\) turns cofiltered limits into filtered colimits.

\[
\text{Hom}(\lim_{X \in U} \prod_{i \in X} M_i, [n]) \cong \lim_{X \in U} \text{Hom}(\prod_{i \in X} M_i, [n]) \\
\cong \lim_{X \in U} \prod_{i \in X} \text{Hom}(M_i, [n])
\]

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Ultracoproducts of costructures

Let \((M_i)_{i \in I}\) be a family of costructures, and let \(U\) be an ultrafilter on \(I\). Define \(\bigsqcup_U M_i = \lim_{\leftarrow X \in U} \coprod_{i \in X} M_i\) (limits and coproducts taken in Stone).

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\]

So an \(n\)-cotuple \(A : \bigsqcup_U M_i \to [n]\) is determined by an \(n\)-cotuple \(A_i : M_i \to [n]\) for each \(i\), modulo equality on a set in the ultrafilter.

Define \(\bigsqcup_U M_i \models R(A)\) iff \(\{i \in I \mid M_i \models R(A_i)\} \in U\).
Łos’s theorem

Let $PS(M)$ be the presheaf structure corresponding to the costructure $M$.

We have $PS(\bigsqcup_{i \in I} M_i) \cong \prod_{i \in I} PS(M_i)$.

Using duality, or adapting the usual proof:

**Theorem (Łos’s theorem for cologic)**

Let $U$ be an ultrafilter on $I$, and let $\{M_i \mid i \in I\}$ be a family of costructures. Let $\varphi(X)$ be a coformula in context $X$, let $A : \bigsqcup_{i \in I} M_i \to X$ be an $X$-cotuple from $M$, and let $(A_i : M_i \to X)_{i \in I}$ be any lift of $A$ to a family of $X$-cotuples from the $M_i$. Then $\bigsqcup_{i \in I} M_i \models \varphi(A)$ if and only if $\{i \in I \mid M_i \models \varphi(A_i)\} \in U$.

Łos’s theorem also gives a direct proof of the compactness theorem for cologic, in the usual way.
Motivation 4: Coalgebraic logic

[Rutten, Adámek, Kurz, Rosický, Moss, etc.]
Motivation 4: Coalgebraic logic

[Rutten, Adámek, Kurz, Rosický, Moss, etc.]

Let $C$ be a category, and let $F$ be a functor $C \to C$.

An $F$-algebra is an object $A$ and a map $\eta: F(A) \to A$.

**Example:** $F: \text{Set} \to \text{Set}, \; F(X) = X^2 \sqcup C$ ($C$ a set).

An $F$-algebra is a set $A$ and a map $\eta: A^2 \sqcup C \to A$, determined by maps $f: A^2 \to A$ and $c: C \to A$, i.e. a structure for the signature with one binary function symbol $f$ and a set $C$ of constant symbols.
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An $F$-coalgebra is an object $A$ and a map $\varepsilon: A \to F(A)$.

**Example:** $F: \text{Stone} \to \text{Stone}, \; F(X) = X^2 \times C$ ($C$ a Stone space).

An $F$-coalgebra is a Stone space $S$ and a map $\eta: S \to S^2 \times C$, determined by maps $f: S \to S$, $g: S \to S$, and $c: S \to C$. 
The constant space $C$ could be any Stone space, e.g. the underlying space of a costructure. For this example, we’ll take $C = \{a, b\}$.

A coalgebra for the functor $F(X) = X^2 \times C$ is an transition system with inputs $f$ and $g$, labeled by the constants $C$. 

There are notions of “universal coalgebra” and “varieties of coalgebras” for coalgebras on Set, dual to classical universal algebra. But the “coalgebraic logics” in these frameworks are infinitary. Cologic gives a finitary compact logic for coalgebras on Stone.
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Question: What is a cotuple of coterms?
Answer: A cotuple from the coterm coalgebra.

Question: What is the coterm coalgebra?
Answer: The cofree coalgebra on the covariables.

\[
X = \{x_1, x_2\} \quad \text{(for example)}
\]

A clopen set from the cofree coalgebra corresponds to a finite partial description of such a behavior:

• \((a, x_1) \underset{f}{\rightarrow} \underset{g}{\rightarrow} \) 

• \((b, x_1 \sqcup x_2) \)

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**Joke**

**Question:** What is a cotuple of coterms?

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**Question:** What is the coterms coalgebra?

**Answer:** The cofree coalgebra on the covariables.

The cofree coalgebra on the covariables $X = \{x_1, x_2\}$ (for example) contains elements witnessing all possible behaviors under $\{f, g\}$-transitions and labelling by $C$ and $X$ (i.e. complete binary trees labeled by $C \times X$). A clopen set from the cofree coalgebra corresponds to a finite partial description of such a behavior:

```
      \bullet(a,x_1)
         /\   \ \\
        f \  g
         /\  /\ \\
    \bullet(b,x_1\sqcup x_2) \bullet(a\sqcup b,x_2)
```
Coterm

For example, let $C(X)$ be the cofree coalgebra on $X$. There is a 2-coterm (a continuous map $t: C(X) \to [2]$) described by

$$(\bullet(x_1,a) \xrightarrow{f} \bullet(x_2,b), \text{everything else}).$$
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Then the sentence $\forall (x_1 \sqcup x_2 = *) \boxtimes (\bullet(x_1,a) \xrightarrow{f} \bullet(x_2,b), \text{everything else})$ asserts that no clopen partition can separate two points $s$ and $t$, such that $s$ is labeled by $a$, $t$ is labeled by $b$, and $f(s) = t$.

Since any two distinct points can be separated by a clopen partition, this means there are no such points.
For example, let $C(X)$ be the cofree coalgebra on $X$. There is a 2-coterm (a continuous map $t: C(X) \to [2]$) described by

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Since any two distinct points can be separated by a clopen partition, this means there are no such points.

It’s not hard to give a concrete syntax for coterms for simple functors like $F$, but it’s work in progress to extend this to a more class of cofinitary functors on Stone.
Syntax and semantics, this time with coalgebra

Let $C(X)$ be the cofree coalgebra on $X$. An $n$-coterm in context $X$ is an $n$-cotuple from $C(X)$, a continuous map $t: C(X) \to [n]$.

An atomic coformula in context $X$ is:

- $R(t(X))$, where $t$ is an $\ar(R)$-coterm in context $X$.
- $\boxtimes_i t(X)$, where $t$ is an $n$-coterm in context $X$ and $1 \leq i \leq n$.

Let $M$ be a costructure given together with an $X$-cotuple $A: M \to X$. $A$ induces a canonical map $A': M \to C(X)$, and any $n$-coterm in context $X$, $t: C(X) \to [n]$, induces an $n$-cotuple $t \circ A'$, which we denote $t(A)$.

\[
\begin{aligned}
&M \xrightarrow{A'} C(X) \xrightarrow{t} [n] \\
&\downarrow A \quad \downarrow \pi \\
&X
\end{aligned}
\]

$M \models R(t(A))$ if and only if $t(A) \in R^M$.

$M \models \boxtimes_i t(A)$ if and only if $t(A)^{-1}(\{i\}) = \emptyset$. 

Future plans

- Cologic on general pro-categories (& logic on general ind-categories!).
- Syntax for terms for general cofinitary functors.
- Develop more model theory, e.g. stability theory.
- Consider more general compact Hausdorff costructures (as suggested by the work of Panagiotopoulos and Bankston).
- Explore possible connections to:
  - Applications of coalgebras, e.g. in modal logic
  - Dual Ramsey theory (via “coindiscernibles”?)
- Try to combine ordinary logic and cologic into a compact second-order logic, via an $\in_{i,j}$ relation between tuples $a : [m] \rightarrow M$ and cotuples $A : M \rightarrow [n]$:
  \[
  M \models a \in_{i,j} A \iff A(a(i)) = j \text{ (i.e. } a_i \in A_j). \]