Independence in generic expansions and fusions

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University of Maryland Logic Seminar
February 20, 2018
I’ll be telling you about the content of two papers:

- **Generic expansion and Skolemization in NSOP$_1$ theories** (with Nicholas Ramsey), arXiv:1706.06616, June 2017.
- **Interpolative fusions II: preservation results**, in preparation.

Outline:

1. **Background**
   - Notions of independence
   - Forking independence in simple theories
   - Kim independence in NSOP$_1$ theories

2. **Example: The generic theory of $\mathcal{L}$-structures**

3. **NSOP$_1$ preservation results**
   - Old results for simple theories
   - Generic expansions and generic Skolemizations
   - Interpolative fusions
Notions of independence

A major theme in model theory is the identification of abstract notions of independence in models of first-order theories.

Fix a complete first-order theory $T$ and a large highly saturated and homogeneous “monster model” $M \models T$. By convention:

- All small models are elementary substructures $M \prec M$.
- All small tuples $b$ are tuples from $M$.
- All small sets are subsets of $M$.

Small means size at most $\kappa$, where $M$ is $\kappa^+$-saturated.

A “notion of independence” is presented as a ternary relation $\mathcal{I}$ on subsets of $M$. For any small sets $A, B, C$, we read $A \mathcal{I} C B$ as "$A$ is independent from $B$ over $C$".
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A “notion of independence” is presented as a ternary relation $\downarrow$ on subsets of $\mathbb{M}$. For any small sets $A, B, C$, we read

$$A \downarrow \frac{B}{C} \text{ as “} A \text{ is independent from } B \text{ over } C \text{”.}$$
Algebraic independence ($\downarrow^a$)

One of the most basic examples is algebraic independence.

**Definition**

A formula $\varphi(x; a)$ is *algebraic* if it has only finitely many solutions. The *algebraic closure* of $A$, $\text{acl}(A)$, is the set of all elements $b \in \mathbb{M}$ which satisfy some algebraic formula with parameters from $A$.

Define $A \downarrow^a_C B \iff \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$. 
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Define \( A \Downarrow^a C B \iff \text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C) \).

In any theory, \( \Downarrow^a \) always satisfies some basic properties:

- **Invariance**: If \( A \Downarrow^a C B \) and \( A'B'C' \equiv ABC \), then \( A' \Downarrow^a C' B' \).
- **Symmetry**: If \( A \Downarrow^a C B \), then \( B \Downarrow^a C A \).
- **Monotonicity**: If \( A' \subseteq A \), \( B' \subseteq B \), and \( A \Downarrow^a C B \), then \( A' \Downarrow^a C B' \).
- **Existence**: \( A \Downarrow^a C C \).
- **Extension**: If \( A \Downarrow^a C B \), and \( B \subseteq B' \), then there exists \( A' \equiv_{BC} A \) such that \( A' \Downarrow^a C B' \).
Definition (Shelah)

A formula $\varphi(x; b)$ divides over $C$ if there is a $C$-indiscernible sequence $(b_i)_{i \in \omega}$ with $b_0 = b$ such that $\{\varphi(x; b_i) \mid i \in \omega\}$ is inconsistent.

Define $A \downarrow^d_C B \iff$ no formula in $\text{tp}(A/BC)$ divides over $C$. 
Dividing independence ($\downarrow^d$) and forking independence ($\downarrow^f$)

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$\downarrow^d$ may not satisfy extension. This motivates the following definition:

**Definition (Shelah)**

A formula $\varphi(x; b)$ *forks* over $C$ if it implies a disjunction $\bigvee_{i=1}^n \psi_i(x; b_i)$ such that each formula $\psi_i(x; b_i)$ divides over $C$.

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Define $A \downarrow^f_C B \iff$ no formula in $\text{tp}(A/BC)$ forks over $C$.

Equivalently, $\downarrow^f$ can be defined by “forcing” extension on $\downarrow^d$:

$A \downarrow^f_C B \iff$ for all $B' \supseteq B$, there is $A' \equiv_{BC} A$ such that $A' \downarrow^d_C B'$. 

Alex Kruckman (IU Bloomington)
Simple theories are the “natural habitat” of forking independence.

**Definition (Shelah ’80)**

A formula $\varphi(x; y)$ has the *tree property* (TP) if there exist tuples $(a_\eta)_{\eta \in \omega^< \omega}$ and $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x; a_\sigma|_n) \mid n \in \omega\}$ is consistent, but for any $\eta \in \omega^< \omega$, $\{\varphi(x; a_\eta|_n) \mid n \in \omega\}$ is $k$-inconsistent (meaning that any subset of size $k$ is inconsistent).

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**Theorem (Kim ’96)**

- $T$ is simple if and only if $\Downarrow^f$ is symmetric: $A \Downarrow^f_C B \iff B \Downarrow^f_C A$.
- If $T$ is simple, then $\Downarrow^f = \Downarrow^d$. 

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Characterizing $\mathcal{L}$

**Theorem (Kim–Pillay)**

Let $T$ be a complete theory and $\mathcal{L}$ any ternary relation on subsets of $\mathbb{M}$. Then $T$ is simple and $\mathcal{L} = \mathcal{L}^f$ if and only if $\mathcal{L}$ satisfies:

- **Invariance.**
- **Symmetry.**
- **Existence.**
- **Extension.**
- **Base monotonicity:** If $D \subseteq C \subseteq B$ and $A \mathcal{L} D B$, then $A \mathcal{L} C B$.
- **Right transitivity:** If $D \subseteq C \subseteq B$, $A \mathcal{L} D C$, and $A \mathcal{L} C B$, then $A \mathcal{L} D B$.
- **Finite character:** $A \mathcal{L} C B$ iff for every finite $B' \subseteq B$, $A \mathcal{L} C B'$.
- **Local character:** For all finite $A$ and all $B$, there is $C \subseteq B$ such that $|C| \leq |T|$ and $A \mathcal{L} C B$.

... and the independence theorem: see next slide.
The independence theorem

Let $M \prec \mathbb{M}$ be a model, $A$ and $B$ sets, and $a$ and $a'$ tuples. Suppose:

1. $a \equiv_M a'$,
2. $A \downarrow_M B$,
3. $a \downarrow_M A$, and
4. $a' \downarrow_M B$.

Then there exists $a''$ such that:

1. $a'' \equiv_{MA} a$,
2. $a'' \equiv_{MB} a'$, and
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2. $a'' \equiv_M B a'$, and
3. $a'' \downarrow_M AB$.
Stable theories

Forking independence was initially introduced to study stable theories.

**Definition (Shelah)**

A formula $\varphi(x; y)$ has the *order property* (OP) if there exist tuples $(a_i)_{i \in \omega}$ and $(b_j)_{j \in \omega}$ such that $M \models \varphi(a_i; b_j)$ if and only if $i \leq j$.

$T$ is *stable* if no formula has OP.

**Theorem**

A simple theory $T$ is stable iff $\upharpoonright^f$ satisfies stationarity:

If $C = acl^{eq}(C')$ and $C \subseteq B$, then any type over $C$ has a unique extension to a type over $B$ which does not fork over $C$. 
A map of the (first-order) universe

source: forkinganddividing.com
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Definition

A global type $p(y) \in S_y(\mathcal{M})$ is $M$-invariant if for all formulas $\psi(y; z)$ and all $c \equiv_M c'$, $\psi(y; c) \in p \iff \psi(y; c') \in p$. 
Definition

A global type \( p(y) \in S_y(\mathbb{M}) \) is \( M \)-invariant if for all formulas \( \psi(y; z) \) and all \( c \equiv_M c' \), \( \psi(y; c) \in p \iff \psi(y; c') \in p \).

Definition

If \( p(y) \in S_y(\mathbb{M}) \) is \( M \)-invariant, a \( p \)-Morley sequence over \( M \) is a sequence \( (b_i)_{i \in \omega} \) such that \( b_i \models p(y)|_{Mb_0...b_{i-1}} \) for all \( i \).

Fact: Any \( p \)-Morley sequence over \( M \) is \( M \)-indiscernible.
Kim dividing and Kim independence ($\downarrow^K$)

**Definition**
A global type $p(y) \in S_y(M)$ is $M$-invariant if for all formulas $\psi(y; z)$ and all $c \equiv_M c'$, $\psi(y; c) \in p \iff \psi(y; c') \in p$.

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**Fact:** Any $p$-Morley sequence over $M$ is $M$-indiscernible.

**Definition (Ramsey, after a suggestion of Kim)**
A formula $\varphi(x; b)$ Kim divides over $M$ if there is a global $M$-invariant type $p(y)$ extending $tp(b/M)$ and a $p$-Morley sequence $(b_i)_{i \in \omega}$ over $M$ such that $\{ \varphi(x, b_i) \mid i \in \omega \}$ is inconsistent.

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A formula $\varphi(x; b)$ **Kim forks** over $M$ if it implies a disjunction of Kim dividing formulas.

Define $a \Downarrow^K_M b \iff$ no formula in $\text{tp}(a/Mb)$ Kim forks over $M$. 
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We only define Kim independence over $M$ when $M$ is a model. Why?

**Fact:** If $M \prec \mathbb{M}$, then every type $q(y) \in \text{S}_y(M)$ extends to a global $M$-invariant type (e.g. any coheir extension).

This fails over an arbitrary set $A$. 
Example: Generic binary functions

Let $\mathcal{L} = \{ f \}$, the language with a single binary function symbol.

**Fact:** The empty $\mathcal{L}$-theory has a model companion, $T^\emptyset_\mathcal{L}$.

$T^\emptyset_\mathcal{L}$ is the “generic theory of binary functions”, or the “theory of existentially closed magmas”.

Consider the formula $\phi(x; y, z) : f(x, y) = z$.

If $b \not\in M$ and $c \in acl(Mb)$, then $\phi(x; b, c)$ divides over $M$.

Let $(b_i, c_i)_{i \in \omega}$ be an $M$-indiscernible sequence such that $b_i = b$ for all $i$ but the $c_i$ are distinct. Then $\{ f(x, b_i) = c_i | i \in \omega \}$ is inconsistent.

But $\phi(x; b, c)$ does not Kim divide over $M$.

If $(b_i, c_i)_{i \in \omega}$ is a Morley sequence over $M$, then the $b_i$ are all distinct, and $\{ f(x, b_i) = c_i | i \in \omega \}$ is consistent.
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Consider the formula $\varphi(x; y, z) : f(x, y) = z$.

- If $b \notin M$ and $c \notin \text{acl}(Mb)$, then $\varphi(x; b, c)$ divides over $M$:
  Let $(b_i c_i)_{i \in \omega}$ be an $M$-indiscernible sequence such that $b_i = b$ for all $i$ but the $c_i$ are distinct. Then $\{f(x, b) = c_i \mid i \in \omega\}$ is inconsistent.
- But $\varphi(x; b, c)$ does not Kim divide over $M$:
  If $(b_i c_i)_{i \in \omega}$ is a Morley sequence over $M$, then the $b_i$ are all distinct, and $\{f(x, b_i) = c_i \mid i \in \omega\}$ is consistent.
In fact, $\varphi(x; y, z)$ has $\text{TP}_2 \implies \text{TP}$, so $T^\emptyset_L$ is not simple.

**Definition**

A formula $\varphi(x; y)$ has the *tree property* 2 (TP$_2$) if there exist tuples $(a_i, j)_{i, j \in \omega}$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x; a_n, \sigma(n)) \mid n < \omega\}$ is consistent, but for any $n < \omega$ and $i < j < \omega$, $\{\varphi(x; a_n, i), \varphi(x; a_n, j)\}$ is inconsistent. $T$ is NTP$_2$ if no formula has TP$_2$. 

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In fact, $\varphi(x; y, z)$ has $\text{TP}_2 \implies \text{TP}$, so $T_L^\emptyset$ is not simple.

**Definition**

A formula $\varphi(x; y)$ has the tree property 2 (TP$_2$) if there exist tuples $(a_{i, j})_{i, j < \omega}$ such that for all $\sigma \in \omega^\omega$, \(\{ \varphi(x; a_{n, \sigma(n)}) \mid n < \omega \}\) is consistent, but for any $n < \omega$ and $i < j < \omega$, \(\{ \varphi(x; a_{n, i}), \varphi(x; a_{n, j}) \}\) is inconsistent. $T$ is NTP$_2$ if no formula has TP$_2$.

Let $(b_i)_{i < \omega}$ and $(c_{i, j})_{i, j < \omega}$ be distinct, and set $a_{i, j} = (b_i, c_{i, j})$.

- $\{ f(x, b_n) = c_{n, \sigma(n)} \mid n < \omega \}$ is consistent, while
- $\{ f(x, b_n) = c_{n, i}, f(x, b_n) = c_{n, j} \}$ is inconsistent.

**Fact:** In NTP$_2$ theories, dividing and Kim dividing agree over models.
NSOP$_1$ theories are the “natural habitat” of Kim independence.

**Definition (Shelah ’04)**

A formula $\varphi(x; y)$ has SOP$_1$ if there exist tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x; a_\sigma|_n) \mid n \in \omega\}$ is consistent, but for any $\nu, \eta \in 2^{<\omega}$, if $\nu^\leq 0 \leq \eta$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu^\leq 1)\}$ is inconsistent.

$T$ is NSOP$_1$ if no formula has SOP$_1$. 
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**Definition (Shelah ’04)**

A formula $\varphi(x;y)$ has SOP$_1$ if there exist tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ such that for all $\sigma \in 2^{\omega}$, $\{\varphi(x; a_{\sigma\mid n}) \mid n \in \omega\}$ is consistent, but for any $\nu, \eta \in 2^{<\omega}$, if $\nu\upharpoonright 0 \leq \eta$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu\uparrow 1)\}$ is inconsistent.

$T$ is NSOP$_1$ if no formula has SOP$_1$.

**Theorem (Kaplan–Ramsey ’17)**

- $T$ is NSOP$_1$ if and only if $\mathrel{\downarrow^K}$ is symmetric.
- If $T$ is NSOP$_1$, Kim forking equals Kim dividing.
- An NSOP$_1$ theory $T$ is simple iff $\mathrel{\mathrel{\downarrow^f}_M \Leftrightarrow \downarrow^K_M}$ (in any theory, $\mathrel{\mathrel{\downarrow^f}_M \Rightarrow \downarrow^K_M}$).
Theorem (Kaplan–Ramsey ’17)

Let $T$ be a complete theory and $\downarrow$ any ternary relation on subsets of $M$. Then $T$ is NSOP$_1$ and $\downarrow_M = \downarrow^K_M$ for all $M < \mathbb{M}$ if and only if $\downarrow_M$ satisfies:

1. **Invariance**: If $A \downarrow_M B$ and $A' B' M' \equiv ABM$, then $A' \downarrow_M B'$.
2. **Symmetry**: If $A \downarrow_M B$, then $B \downarrow_M A$.
3. **Monotonicity**: If $A' \subseteq A$, $B' \subseteq B$, and $A \downarrow_M B$, then $A' \downarrow_M B'$.
4. **Existence**: $A \downarrow_M M$.
5. **Strong finite character and witnessing**: if $A \not\downarrow_M B$, then there is a formula $\varphi(x; b) \in \text{tp}(A/MB)$ such that for any $a' \models \varphi(x; b)$, $a' \not\downarrow_M b$. Moreover, $\varphi(x; b)$ Kim divides over $M$.
6. **The independence theorem**.
Using the new criterion for NSOP$_1$, all known examples of NSOP$_3$ theories have been shown to be NSOP$_1$ (Chernikov, Conant, K., Ramsey, others).
Fact: In any language $\mathcal{L}$, the empty $\mathcal{L}$-theory has a model companion $T^\emptyset_\mathcal{L}$, the theory of existentially closed $\mathcal{L}$-structures.

We call $T^\emptyset_\mathcal{L}$ the \textit{generic theory of $\mathcal{L}$-structures}.

When $\mathcal{L}$ contains constant symbols, $T^\emptyset_\mathcal{L}$ is not complete. But the completions of $T^\emptyset_\mathcal{L}$ are classified by the isomorphism type of $\langle \emptyset \rangle$.

If $\mathcal{L}$ is relational, then $T^\emptyset_\mathcal{L}$ is simple (and $\aleph_0$-categorical).

But if $\mathcal{L}$ contains an $n$-ary function, $n \geq 2$, then $T^\emptyset_\mathcal{L}$ has TP$_2$.

Jeřábek showed that $T^\emptyset_\mathcal{L}$ is NSOP$_3$ for any language $\mathcal{L}$, and he asked if it is NSOP$_1$. (Later, he independently answered this question.)
Theorem (K.–Ramsey, independently Jeřábek)

For any language $\mathcal{L}$:

- $T^\emptyset_\mathcal{L}$ eliminates quantifiers, and $\text{acl}(A) = \langle A \rangle$, the substructure generated by $A$.
- $\models^a$ satisfies the independence theorem over arbitrary sets.
- It follows easily that $T^\emptyset_\mathcal{L}$ is NSOP$_1$ and $\models^K = \models^a$ over models.
Theorem (K.–Ramsey, independently Jeřábek)

For any language $\mathcal{L}$:

- $T^\emptyset_{\mathcal{L}}$ eliminates quantifiers, and $\text{acl}(A) = \langle A \rangle$, the substructure generated by $A$.
- $\downarrow^a$ satisfies the independence theorem over arbitrary sets.
- It follows easily that $T^\emptyset_{\mathcal{L}}$ is NSOP$_1$ and $\downarrow^K = \downarrow^a$ over models.

Jeřábek’s preprint contains a complete classification of $T^\emptyset_{\mathcal{L}}$:

<table>
<thead>
<tr>
<th>Relation arities:</th>
<th>$\leq 0$</th>
<th>$\leq 1$</th>
<th>any</th>
<th>any</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function arities:</td>
<td>$\leq 0$</td>
<td>$\leq 1$</td>
<td>$\leq 1$</td>
<td>any</td>
</tr>
</tbody>
</table>

$T^\emptyset_{\mathcal{L}}$ is:

- strongly minimal
- stable*
- simple*
- NSOP$_1$

* If $T^\emptyset_{\mathcal{L}}$ is stable/simple, then it is superstable/supersimple if and only if there is at most one unary function symbol in $\mathcal{L}$. 

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* Alex Kruckman (IU Bloomington)  Generic expansions and fusions  UMD Logic Seminar 2/20/18 18 / 35
Recall that $\downarrow^f$ satisfies base monotonicity in any theory.

Base monotonicity: If $D \subseteq C \subseteq B$ and $A \downarrow_D B$, then $A \downarrow_C B$.

But $\downarrow^K$ and $\downarrow^\alpha$ lack base monotonicity in general.
Dividing and base monotonicity

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Base monotonicity: If $D \subseteq C \subseteq B$ and $A \mathrel{\downarrow_D} B$, then $A \mathrel{\downarrow_C} B$.

But $\mathrel{\downarrow^K}$ and $\mathrel{\downarrow^a}$ lack base monotonicity in general.

**Theorem (K.–Ramsey)**

In $T_{\mathcal{L}}^{\emptyset}$, $\mathrel{\downarrow^f} = \mathrel{\downarrow^d}$, and this relation is obtained by “forcing” base monotonicity on $\mathrel{\downarrow^a}$:

$$A \mathrel{\downarrow^d_C} B \text{ if and only if } A \mathrel{\downarrow^a_C} B \text{ for all } C \subseteq C' \subseteq \acl(BC).$$

For example, generically, if $f(a, b) = c$, then:

- $bc \mathrel{\downarrow^a} a$, and $bc \mathrel{\downarrow^f} a$
- $a \mathrel{\downarrow^a} bc$, but $a \mathrel{\downarrow^a_b} bc$, so $a \mathrel{\downarrow^f} bc$. 
Preservation results: Adding generic structure

Recipe:

1. Start with a base $\mathcal{L}$-theory $T$.
2. Add new symbols: $\mathcal{L} \subseteq \mathcal{L}_{\text{new}}$.
3. And new axioms governing them: $T \subseteq T_{\text{new}}$.
4. Take the model companion (if it exists): $T_{\text{new}}^*$.

Example 0: Generic automorphisms

$L_{\text{new}} = L \cup \{\sigma\}$, a unary function symbol.

$T_{\text{new}} = T \cup \text{"\sigma is an } L \text{-automorphism"}$.

$T_{\text{new}}^* = T_{A}$, the theory $T$ with a generic automorphism [e.g. if $T = \text{ACF}$, then $T_{A} = \text{ACFA}$]

The question of whether $T_{A}$ exists is often nontrivial.
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- $\mathcal{L}_{\text{new}} = \mathcal{L} \cup \{\sigma\}$, a unary function symbol.
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Example 1: Generic expansions

- $\mathcal{L}_{\text{new}} = \mathcal{L}'$, any expansion of $\mathcal{L}$ by new constant, function, and relation symbols.
- $T_{\text{new}} = T$, so the new symbols are interpreted arbitrarily.
- $T^*_{\text{new}} = T_{\mathcal{L}'}$, the generic expansion of $T$ to $\mathcal{L}'$.

Consider the special case $\mathcal{L} = \emptyset$ and $T = \emptyset$. Then $T_{\mathcal{L}'}$ is just the generic theory of $\mathcal{L}'$-structures.
Example 1: Generic expansions

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Consider the special case $\mathcal{L} = \emptyset$ and $T = \emptyset$. Then $T_{\mathcal{L}'}$ is just the generic theory of $\mathcal{L}'$-structures.

**Theorem (Winkler '75)**

*If $T$ is model complete and eliminates $\exists^\infty$, then $T_{\mathcal{L}'}$ exists.*
Definition

A definable function $f_\varphi(y)$ is a *Skolem function* for the formula $\varphi(x; y)$ if $\mathcal{M} \models \varphi(f_\varphi(\bar{a}), \bar{a})$ whenever $\varphi(\mathcal{M}, \bar{a})$ is nonempty.
Generic Skolemizations

**Definition**

A definable function $f_\varphi(y)$ is a *Skolem function* for the formula $\varphi(x; y)$ if $M \models \varphi(f_\varphi(a), a)$ whenever $\varphi(M, a)$ is nonempty.

**Example 2: Generic Skolemizations**

1. $L_{\text{new}} = L_{\text{Sk}} = L \cup \{f_\varphi \mid \varphi(x; y) \text{ an } L\text{-formula}\}$.
2. $T_{\text{new}} = T \cup \{\forall y (\exists x \varphi(x; y) \rightarrow \varphi(f_\varphi(y); y)) \mid \varphi(x; y) \text{ an } L\text{-formula}\}$.
3. $T^*_{\text{new}} = T_{\text{Sk}}$, the generic Skolemization of $T$. 

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- $T^*_{\text{new}} = T_{\text{Sk}}$, the generic Skolemization of $T$.

### Theorem (Winkler ’75)

*If $T$ is model complete and eliminates $\exists^\infty$, then $T_{\text{Sk}}$ exists.*
For the rest of this talk, assume $T$ is model complete and eliminates $\exists^\infty$. 
Preserving simplicity

For the rest of this talk, assume $T$ is model complete and eliminates $\exists^\infty$.

**Theorem (Chatzidakis–Pillay ’98)**

- If $T$ is stable and $T_A$ exists, then $T_A$ is simple.
- If $T$ is simple and $\mathcal{L}' = \mathcal{L} \cup \{P\}$, where $P$ is a unary relation symbol, then $T_{\mathcal{L}'}$ is simple.
Preserving simplicity

For the rest of this talk, assume $T$ is model complete and eliminates $\exists^\infty$.

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- If $T$ is simple and $\mathcal{L}' = \mathcal{L} \cup \{P\}$, where $P$ is a unary relation symbol, then $T_{\mathcal{L}'}$ is simple.

**Theorem (Nübling ’03)**
- Let $\mathcal{L}' = \mathcal{L} \cup \{f\}$, where $f$ is a unary function symbol. If $T$ is simple with QE, and $\text{acl}(A) = A$ for all sets $A$, then $T_{\mathcal{L}'}$ is simple.
- Let $T^a_{Sk}$ be the theory obtained by adding generic Skolem functions for algebraic formulas only. If $T$ is simple, then $T^a_{Sk}$ is simple.
Preserving simplicity

Each of the results on the previous slide (except generic expansion by a unary function, which Nübling proved by counting types), has the following proof strategy:

- Let $\mathcal{M}' \models T^*_\text{new}$ be a monster model and $\mathcal{M} \models T$ its reduct to $\mathcal{L}$.
- Define a notion of independence in $\mathcal{M}'$ in terms of $\downarrow_f$ in $\mathcal{M}$.

$$a \downarrow_C b \text{ in } \mathcal{M}' \iff \text{acl}_{T^*_\text{new}}(C a) \downarrow_C \text{acl}_{T^*_\text{new}}(C b) \text{ in } \mathcal{M}.$$ 

- Apply the Kim–Pillay theorem characterizing $\downarrow_f$ in simple theories. The main difficulty is checking the independence theorem.
Preserving $\text{NSOP}_1$

**Theorem (K.–Ramsey)**

- For any $\mathcal{L}' \supseteq \mathcal{L}$, if $T$ is $\text{NSOP}_1$, then $T_{\mathcal{L}'}$ is $\text{NSOP}_1$.
- If $T$ is $\text{NSOP}_1$, then $T_{\text{Sk}}$ is $\text{NSOP}_1$.

**Proof strategy:**

- Define a notion of independence in $\mathcal{M}'$ in terms of $\downarrow^K$ in $\mathcal{M}$.

  \[
  a \downarrow^K b \text{ in } \mathcal{M}' \iff \text{acl}_{T_{\text{new}}}^* (Ca) \downarrow^K_{C} \text{acl}_{T_{\text{new}}}^* (Cb) \text{ in } \mathcal{M}.
  \]

- Apply the Kaplan–Ramsey theorem characterizing $\downarrow^K$ in $\text{NSOP}_1$. Again, the main difficulty is the independence theorem.
The generic Skolemization $T_{Sk}$ has a Skolem function for every $\mathcal{L}$-formula, but not necessarily for every $\mathcal{L}_{Sk}$-formula. But we can iterate the construction to obtain an expansion with Skolem functions for all formulas.

**Corollary (K.–Ramsey)**

Any $NSOP_1$ theory $T$ which eliminates $\exists^\infty$ has an expansion to an $NSOP_1$ theory with built-in Skolem functions.
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**Corollary (K.–Ramsey)**

Any NSOP$_1$ theory $T$ which eliminates $\exists^\infty$ has an expansion to an NSOP$_1$ theory with built-in Skolem functions.

This result may turn out to be a useful technical tool: in an NSOP$_1$ theory with built-in Skolem functions, $\downarrow^K$ makes sense over an arbitrary base $C$, since $\text{dcl}(C)$ is a model.
Theorem (Winkler ’75)

If $T_1$ and $T_2$ are theories in disjoint languages, and each is model complete and eliminates $\exists^\infty$, then $T_1 \cup T_2$ has a model companion $T$.  

Example:

$T_1$ is the theory of divisible abelian groups in the language $\{0, +, -\}$.

$T_2$ is the theory of an equivalence relation with infinitely many infinite classes in the language $\{E\}$.

$T_1$ and $T_2$ are both stable, but $\phi(x; y, z) : (x + y) E z$ has TP in $T$. 

Let $(v_i)_{i \in \omega}$ be distinct, let $(e_j)_{j \in \omega}$ be representatives of distinct equivalence classes, and set $a_{i,j} = (v_i, e_j)$.

$\{(x + v_n) E e_{\sigma(n)} | n < \omega\}$ is consistent, while $\{(x + v_n) E e_i, (x + v_n) E e_j\}$ is inconsistent.
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If $T_1$ and $T_2$ are NSOP$_1$, is $T$ NSOP$_1$?
Interpolative fusions

If $T_1$ and $T_2$ are NSOP$_1$, is $T$ NSOP$_1$?

Yes! But we’ll prove this in a more general context:

1. $\mathcal{L}_1$ and $\mathcal{L}_2$ are first-order languages with intersection $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$ and union $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$.

2. $T_1$ is a complete $\mathcal{L}_1$-theory, and $T_2$ is a complete $\mathcal{L}_2$-theory. So $T_0 = T_1 \cap T_2$ is a complete $\mathcal{L}_0$-theory.

3. $T_0$, $T_1$, and $T_2$ all have quantifier elimination.
Interpolative fusions

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- $T_1$ is a complete $\mathcal{L}_1$-theory, and $T_2$ is a complete $\mathcal{L}_2$-theory. So $T_0 = T_1 \cap T_2$ is a complete $\mathcal{L}_0$-theory.
- $T_0$, $T_1$, and $T_2$ all have quantifier elimination.

If the $\mathcal{L}$-theory $T_1 \cup T_2$ has a model companion $T$, we call $T$ the \textit{interpolative fusion} of $T_1$ and $T_2$.

Minh Chieu Tran and Erik Walsberg initiated the study of the interpolative fusion (\textit{Interpolative Fusions I}, in preparation). They provide some sufficient conditions for its existence - we’ll ignore this issue here.
Analysis of the interpolative fusion

From now on, we additionally assume:

- The interpolative fusion $T$ of $T_1$ and $T_2$ over $T_0$ exists.
- $T_0$ is stable with weak elimination of imaginaries (this gives stationarity for $\mathcal{L}$ over all $C = \text{acl}(C')$).
Analysis of the interpolative fusion

From now on, we additionally assume:

- The interpolative fusion $T$ of $T_1$ and $T_2$ over $T_0$ exists.
- $T_0$ is stable with weak elimination of imaginaries (this gives stationarity for $\overline{f}$ over all $C = acl(C')$).

For any $A \subseteq M \models T$, let $cl(A)$ be the least set containing $A$ which is algebraically closed in both $M_1 = M|L_1$ and $M_2 = M|L_2$. 
From now on, we additionally assume:

- The interpolative fusion $T$ of $T_1$ and $T_2$ over $T_0$ exists.
- $T_0$ is stable with weak elimination of imaginaries (this gives stationarity for $\downarrow f$ over all $C = \acl(C)$).

For any $A \subseteq M \models T$, let $\cl(A)$ be the least set containing $A$ which is algebraically closed in both $M_1 = M|L_1$ and $M_2 = M|L_2$.

**Theorem (K.)**

*The category of closed substructures of models of $T$ and embeddings has the disjoint amalgamation property.*

It follows that:

1. $\acl = \cl$ in $T$.
2. For any $a$, $tp(a)$ is determined by $qftp(\cl(a))$ ("Almost QE").
3. The completions of $T$ are classified by the isomorphism type of $\cl(\emptyset)$. 
Proof sketch: consistent amalgamation

The key fact is the following lemma:

**Lemma (K.)**

Fix $i = 1$ or 2. Let $A$ be algebraically closed in $\mathbb{M}_i$, and let $p(x)$ and $q(y)$ be two $\mathcal{L}_i$-types over $A$. Then there are realizations $a \models p(x)$ and $b \models q(y)$ such that $a \frown_A b$ in $\mathbb{M}_0$.

(For this lemma, it suffices that $T_0$ is simple with stable forking and geometric elimination of imaginaries.)
Proof sketch: consistent amalgamation

The key fact is the following lemma:

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Fix $i = 1$ or $2$. Let $A$ be algebraically closed in $\mathbb{M}_i$, and let $p(x)$ and $q(y)$ be two $\mathcal{L}_i$-types over $A$. Then there are realizations $a \models p(x)$ and $b \models q(y)$ such that $a \upharpoonright f_A b$ in $\mathbb{M}_0$.

(For this lemma, it suffices that $T_0$ is simple with stable forking and geometric elimination of imaginaries.)

Then to amalgamate closed subsets of $\mathbb{M}$:

- Separately amalgamate their reducts in $\mathcal{L}_1$ and $\mathcal{L}_2$, using the lemma to make each $\upharpoonright f$-independent in the reduct to $\mathcal{L}_0$.
- The two amalgams are guaranteed to agree in $\mathcal{L}_0$ by stationarity of $\upharpoonright f$, and we can apply the Robinson Joint Consistency Theorem.
Preservation of NSOP$_1$

**Theorem (K.)**

If $T_1$ and $T_2$ are NSOP$_1$ and $T_0$ has 3-uniqueness (in addition to our other hypotheses), then $T$ is NSOP$_1$.

(In particular, this applies to the case of fusions: when $\mathcal{L}_0 = \emptyset$ and $T_0$ is the theory of an infinite set.)
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(In particular, this applies to the case of fusions: when $\mathcal{L}_0 = \emptyset$ and $T_0$ is the theory of an infinite set.)

**Definition**

Suppose $a_1$, $a_2$, and $a_3$ enumerate algebraically closed sets, pairwise $\downarrow^f$-independent over a common algebraically closed subset $A$. For $1 \leq i < j \leq 3$, let $a_{ij}$ be a tuple enumerating $\text{acl}(a_i, a_j)$. $T_0$ has *3-uniqueness* if $\text{tp}(a_{12}) \cup \text{tp}(a_{13}) \cup \text{tp}(a_{23}) \vdash \text{tp}(a_{12}a_{13}a_{23})$.

To get amalgamation in $T$, we assumed weak elimination of imaginaries $\implies$ stationarity = “2-uniqueness” in $T_0$.

To get 3-amalgamation in $T$ (the independence theorem), we assume 3-uniqueness $= \text{elimination of "generalized imaginaries"}$.
To prove the independence theorem in $T$:

- Given $a, a', A, B$, separately apply the independence theorem in the reducts to $\mathcal{L}_1$ and $\mathcal{L}_2$, obtaining $a''$ in each reduct.
- The two amalgams are guaranteed to agree on $\text{acl}_0(a''AB)$ in $\mathcal{L}_0$ by 3-uniqueness.
- To handle the elements which are in $\text{cl}$ but not $\text{acl}_0$, we need a stronger form of the independence theorem which implies that we can take $\text{cl}(a''A) \upharpoonright_{\text{acl}_0(a''AB)} \text{cl}(a''B)$, $\text{cl}(a''A) \upharpoonright_{\text{acl}_0(a''AB)} \text{cl}(AB)$, and $\text{cl}(a''B) \upharpoonright_{\text{acl}_0(a''AB)} \text{cl}(AB)$ in $\mathcal{L}_0$.
- Then we can apply 3-uniqueness again, over $\text{acl}_0(a''AB)$ this time. This implies that the two amalgams agree on all of $\text{cl}(a''AB)$.
- Finally, apply the Robinson Joint Consistency Theorem.
The “reasonable” independence theorem

Let $T_1$ be an NSOP$_1$ theory with a reduct $T_0$ which is simple with stable forking and geometric elimination of imaginaries.

Define $A \fork C B \iff \text{acl}_1(AC) \fork_{\text{acl}_1(C)} \text{acl}_1(BC)$ in $\mathbb{M}_0$.

where $\text{acl}_1$ is algebraic closure in $\mathbb{M}_1$.

Example: If $T_0$ is the theory of an infinite set, then $\fork = \fork^a$.
The “reasonable” independence theorem

Let $T_1$ be an NSOP$_1$ theory with a reduct $T_0$ which is simple with stable forking and geometric elimination of imaginaries.

Define $A \vdash_C B \iff \operatorname{acl}_1(AC) \vdash_{\operatorname{acl}_1(C)} \operatorname{acl}_1(BC)$ in $\mathbb{M}_0$.

where $\operatorname{acl}_1$ is algebraic closure in $\mathbb{M}_1$.

Example: If $T_0$ is the theory of an infinite set, then $\vdash = \vdash^a$.

**Theorem (Independence theorem, Kaplan–Ramsey)**

If $a \vdash^K_M b$, $a' \vdash^K_M c$, $b \vdash^K_M c$, and $a \equiv_M a'$, then there exists $a''$ such that $a'' \equiv_M b$, $a'' \equiv_M c$, and $a'' \vdash^K_M bc$. 
The “reasonable” independence theorem

Let $T_1$ be an NSOP$_1$ theory with a reduct $T_0$ which is simple with stable forking and geometric elimination of imaginaries.

Define $A rown_c B \iff \acl_1(AC) \frown_{\acl_1(C)} \acl_1(BC)$ in $\mathbb{M}_0$.

where $\acl_1$ is algebraic closure in $\mathbb{M}_1$.

Example: If $T_0$ is the theory of an infinite set, then $\frown = \frown^a$.

Theorem (K. ’18, K.–Ramsey ’17 in the case $\frown = \frown^a$)

If $a \downarrow^K_M b$, $a' \downarrow^K_M c$, $b \downarrow^K_M c$, and $a \equiv_M a'$, then there exists $a''$ such that $a'' \equiv_M b$, $a'' \equiv_M c$, and $a'' \downarrow^K_M bc$, and further,

$$a'' \frown c, \quad a'' \frown b, \quad \text{and} \quad b \frown c.$$
Abstract independence without base monotonicity

The previous theorem can be proven replacing $\Downarrow^r$ with any relation $\Downarrow^*$ satisfying:

1. **Invariance**: If $A \Downarrow^*_C B$ and $ABC \equiv A'B'C''$, then $A' \Downarrow^*_{C'} B'$.
2. **Monotonicity**: If $A \Downarrow^*_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \Downarrow^*_C B'$.
3. **Symmetry**: If $A \Downarrow^*_C B$, then $B \Downarrow^*_C A$.
4. **Transitivity**: Suppose $C \subseteq B \subseteq A$. If $A \Downarrow^*_B D$ and $B \Downarrow^*_C D$, then $A \Downarrow^*_C D$.
5. **Normality**: If $A \Downarrow^*_C B$, then $AC \Downarrow^*_C B$.
6. **Full existence**: For any $A$, $B$, and $C$, there exists $A' \equiv_C A$ such that $A' \Downarrow^*_C B$.
7. **Finite character**: If $A' \Downarrow^*_C B$ for all finite $A' \subseteq A$, then $A \Downarrow^*_C B$.
8. **Strong local character**: For every cardinal $\lambda$, there exists a cardinal $\kappa$ such that for all $A$ with $|A| = \lambda$, all $B$, and all $D \subseteq B$, there exists $D \subseteq C \subseteq B$ with $|C| \leq \max(|D|, \kappa)$ and $A \Downarrow^*_C B$. 

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Preservation of simplicity

**Theorem (Kaplan–Ramsey)**

*T is simple if and only if* $T$ *is NSOP*$_1$ *and* $\downarrow^K$ *satisfies base monotonicity over models: for all* $M \prec N \prec \mathbb{M}$, *if* $a \downarrow^K_M N b$, *then* $a \downarrow^K_N b$.

Let $\text{acl}_i$ be algebraic closure computed in $\mathbb{M}_i$.

**Corollary (K.)**

*Suppose* $T_1$ *and* $T_2$ *are simple,* $T_0$ *has 3-uniqueness, and* $\text{cl} = \text{acl}_1 = \text{acl}_2$. *Then* $T$ *is simple.*

**Proof.**

Fix $M \prec N \prec \mathbb{M}$ *and* $a \downarrow^K_M N b$. *Then* $\text{cl}(Ma) \downarrow^K_M \text{cl}(Nb)$ *in* $\mathbb{M}_i$ *for* $i = 1$ *and* 2. *Since* $T_i$ *is simple,* $a \downarrow^f_M N b$ *in* $\mathbb{M}_i$. *Using base monotonicity for* $\downarrow^f$, $a \downarrow^f_N b$, *so* $\text{acl}_i(Na) \downarrow^f_N \text{acl}_i(Nb)$. *Since* $\text{cl} = \text{acl}_i$, $\text{cl}(Na) \downarrow^K_N \text{cl}(Nb)$ *in* $\mathbb{M}_i$. *Thus* $a \downarrow^K_N b$ *in* $\mathbb{M}$, *as desired.*